Unique Fixed Point theorems for Self Mappings Satisfying Contractive Type Conditions in Cone Metric Spaces

Sharad Kumar Barve[*1], Masroor Mohammad[2].

[*1]*Research Scholar Department of Mathematics, J. H. Govt. P. G. College Betul, India-460001.*

[2]*Research Scholar Department of Mathematics Saifia Science College Bhopal, India-462001.*

Abstract:

In this paper, the existence and uniqueness of a fixed point in a cone metric space are discussed for a single self-mapping using expanding and comparison function in the setting of cone metric space. These established results improve and modify some existing results in the literature.

Keywords: Cone metric space, contraction mapping, self-mappings, Cauchy sequence, fixed point.

1 Introduction

Huang and Zhang [13] introduced the concept of acone metric space by replacing the set of real numberswith an ordered Banach space and proved some fixed pointtheorems for mapping satisfying different types of contractiveconditions.Subsequently, many authors (see, e.g. [2], [3],[4],[5],[11],[12])have studied fixed point theorems of Huang and Zhang [13] forcontractive type mappings in cone metric spaces and proved somefixed point and common fixed point theorems in cone metricspaces. Rezapour and Hamlbarani [23] haveobtained some fixed point results in cone metric spaces byomitting the assumption of normality in the results of Huangand Zhang [13].In this Chapter, we proved a unique fixed point theorem incone metric spaces in complete cone metric spaces withoutusing the normality condition. Our result extends andimproves the results of [18] and [19].

2 Preliminaries

Definition 2. 1.[13] Let Ebe a real Banach space and P be a soft subset of$E$. Then $P$is called a cone if and only if

(1) P is closed, $P \neq \varphi$ and $P \neq \{0\}$,

(2) $a, b \in \mathbb{R}; x, y \in P \Longrightarrow ax + by \in P$,

(3) $x \in P$ and $-x \in P$implies $x = \theta$.

Given a cone $P \subset E$, we define a soft partial ordering $\preccurlyeq$ with respect to P by $x \preccurlyeq y$ if and only if $y - x \in p$. We write $x < y$ whenever, $x \preccurlyeq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in Int\ P$ where $Int\ P$ denotes the interior of P. The cone P is called normal if there is a number $k > 0$, such that $\forall x, y \in E$,

we have $\theta \preccurlyeq x \preccurlyeq y \Longrightarrow \| x \| \preccurlyeq k \| y \|$ **2**

The least positive number satisfying this inequality is called the normal constant of P. The cone P is called regular if every increasing sequence which is bounded from above is convergent. Equivalently the cone P is called regular if every decreasing sequence which is bounded from below is convergent. Regular cones are normal and there exist normal cones which are not regular. Throughout the Banach space E and the cone P will be omitted.

Definition 2. 2.[13]Let X be a non-empty set and let $d: X \times X \to E$ be a function satisfying following conditions:

(i) $0 \leq d(x, y)\ \forall x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x) = 0, \Rightarrow x = y$;

(iii) $d(x, y) \leq d(x, z) = d(z, y), \forall x, y, z \in X$.

Then d is called cone metric on X and (X, d) is called cone metric space.

Obviously, the cone metric spaces generalize metric spaces.

Example 2. 3.[13]Let $E = \mathbb{R}^2, P = \{(x, y) \in E: x, y \geq 0\} \subset \mathbb{R}^2, X = \mathbb{R}$ and $d: X \times X \to E$ such that $d(x, y) = (|x - y|, \infty|x - y|)$, where $\infty > 0$ is a constant. Then (X, d) is a cone metric space.

Definition 2. 4.[13] A sequence $\{x_n\}$ in cone metric space (X, d) is called Cauchy sequence if for a given $\varepsilon > 0$, there exists $n_0 \in N$, such that for all $m, n \geq n_0 \Rightarrow d(x_m, x_n) < \varepsilon$ or $d(x_n, x_m) < \varepsilon$ i.e., $min\{d(x_m, x_n), d(x_n, x_m)\} < \varepsilon$.

Definition 2. 5.[13] A sequence $\{x_n\}$ in cone metric space converges to x if

$$\lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} d(x, x_n) = 0$$

In this case, x is called a limit of $\{x_n\}$ and we write $x_n \to x$.

Definition 2. 6.[8]A cone metric space (X, d) is called complete if every Cauchy sequence in it is a convergent sequence.

Definition 2. 7.[13] Let (X, d) be a cone metric space, P be the normal cone with normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X;

If $\{x_n\}$ converges to X and $\{x_n\}$ converges to y, then $x = y$. That is the limit of $\{x_n\}$ is unique.

If $x_n \to x, y_n \to y$ as $n \to \infty$. Then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.

Definition 2. 8.[7]A map $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ is called comparison function if it satisfies:

(i). φ is monotonic increasing;

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(ii). The sequence $\{\varphi^n(t)\}_{n=0}^{\infty}$ converges to zero for all $t \in \mathbb{R}_+$

If φ satisfies:

(iii). $\sum_{k=0}^{\infty} \varphi^k(t)$ converges for all $t \in \mathbb{R}_+$.

Thus every comparison function is $(c) - \text{comparison}$ function. A prototype example for comparison function is $\varphi(t) = \alpha t, t \in \mathbb{R}_+, 0 \le \alpha < 1.$

For further references see [7].

Lemma 2. 9.[7] for $t > 0$ every comparison function implies

$\varphi(t) < t$

and

$\varphi(t) = 0 \; iff \;\; t = 0.$

3Main Results

Theorem 3. 1. Let (X, d)be a complete cone metric space and let $T: X \to X$be a mapping satisfying the following condition.

$$\begin{aligned} d(Tx,Ty) \ge{} & \alpha_1 d(x,y) + \alpha_2[\{d(x,y) + d(Tx,Ty)\}] + \alpha_3[\{d(x,Tx) + d(y,Ty)\}] \\ & + \alpha_4 \frac{[d(Tx,y) + d(x,y)][1 + d(y,Ty)]}{1 + d(x,y)} + \alpha_5 \frac{[d(Tx,y) + d(x,Tx)]d(y,Ty)}{d(x,y)} \\ & + \alpha_6 \frac{d(Tx,y) + d(Tx,Ty) - d(Tx,y)^2 d(Tx,Ty) - d(Tx,y)d(Tx,Ty)^2}{1 - d(Tx,y)d(Tx,Ty)} \end{aligned} \tag{3.1}$$

for all $x, y \in X$; $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \ge 0$; $\alpha_1 + 2(\alpha_2 + \alpha_3) + \alpha_4 + \alpha_5 + \alpha_6 > 1$. Then Thas a unique fixed point.

Proof: Let $\{x_n\}$be a sequence in X, defined as follows

Let $x_0 \in X, x_0 = Tx_1, x_1 = Tx_2, \dots\dots\dots, x_n = Tx_{n+1}$

Consider $d(x_{n-1}, x_n) = d(Tx_n, Tx_{n+1})$

$$\begin{aligned} & \ge \alpha_1 d(x_n,x_{n+1}) + \alpha_2[\{d(x_n,x_{n+1}) + d(Tx_n,Tx_{n+1})\}] + \alpha_3[\{d(x_n,Tx_n) + d(x_{n+1},Tx_{n+1})\}] \\ & + \alpha_4 \frac{[d(Tx_n,x_{n+1}) + d(x_n,x_{n+1})][1 + d(x_{n+1},Tx_{n+1})]}{1 + d(x_n,x_{n+1})} + \beta \frac{[d(Tx_n,x_{n+1}) + d(x_n,Tx_n)]d(x_{n+1},Tx_{n+1})}{d(x_n,x_{n+1})} \\ & + \gamma \frac{d(Tx_n,x_{n+1}) + d(Tx_n,Tx_{n+1}) - d(Tx_n,x_{n+1})^2 d(Tx_n,Tx_{n+1}) - d(Tx_n,x_{n+1})d(Tx_n,Tx_{n+1})^2}{1 - d(Tx_n,x_{n+1})d(Tx_n,Tx_{n+1})} \end{aligned}$$

$$\begin{aligned} = {} & \alpha_1 d(x_n,x_{n+1}) + \alpha_2[\{d(x_n,x_{n+1}) + d(x_{n-1},x_n)\}] + \alpha_3[\{d(x_n,x_{n-1}) + d(x_{n+1},x_n)\}] \\ & + \alpha_4 \frac{[d(x_{n-1},x_{n+1}) + d(x_n,x_{n+1})][1 + d(x_{n+1},x_n)]}{1 + d(x_n,x_{n+1})} \\ & + \alpha_5 \frac{[d(x_{n-1},x_{n+1}) + d(x_n,x_{n-1})]d(x_{n+1},x_n)}{d(x_n,x_{n+1})} \\ & + \alpha_6 \frac{d(x_{n-1},x_{n+1}) + d(x_{n-1},x_n) - d(x_{n-1},x_{n+1})^2 d(x_{n-1},x_n) - d(x_{n-1},x_{n+1})d(x_{n-1},x_n)^2}{1 - d(x_{n-1},x_{n+1})d(x_{n-1},x_n)} \end{aligned}$$

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$$= \alpha_1 d(x_n,x_{n+1}) + \alpha_2[\{d(x_n,x_{n+1}) + d(x_{n-1},x_n)\}] + \alpha_3[\{d(x_n,x_{n-1}) + d(x_{n+1},x_n)\}] + \alpha[d(x_{n-1},x_{n+1}) + d(x_n,x_{n+1})] + \beta[d(x_{n-1},x_{n+1}) + d(x_n,x_{n-1})] + \gamma[d(x_{n-1},x_{n+1}) + d(x_{n-1},x_n)]$$

$$\geq \alpha_1 d(x_n,x_{n+1}) + \alpha_2[\{d(x_n,x_{n+1}) + d(x_{n-1},x_n)\}] + \alpha_3[\{d(x_n,x_{n-1}) + d(x_{n+1},x_n)\}] + \alpha_4 d(x_{n-1},x_n) + \alpha_5 d(x_n,x_{n+1}) + \alpha_6 d(x_n,x_{n+1})$$

$$= (\alpha_2 + \alpha_3 + \alpha_4) d(x_{n-1},x_n) + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6) d(x_n,x_{n+1})$$

$$\Rightarrow [1 - (\alpha_2 + \alpha_3 + \alpha_4)] d(x_{n-1},x_n) \geq (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6) d(x_n,x_{n+1})$$

or $$d(x_n,x_{n+1}) \leq \frac{1 - (\alpha_2 + \alpha_3 + \alpha_4)}{(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 + \alpha_6)} d(x_{n-1},x_n)$$

$d(x_n,x_{n+1}) \leq L d(x_{n-1},x_n)$

Where $\frac{1-(\alpha_2+\alpha_3+\alpha_4)}{(\alpha_1+\alpha_2+\alpha_3+\alpha_5+\alpha_6)} = L, 0 \leq L < 1$

Similarly, we have

$d(x_{n-1},x_n) \leq L d(x_{n-2},x_{n-1})$.

Continuing this process, we conclude that

$d(x_n,x_{n+1}) \leq L^n d(x_0,x_1)$

For $n > m$, using triangular inequality we have

$$d(x_n,x_m) \leq d(x_{n-1},x_n) + d(x_{n-2},x_{n-1}) + \dots\dots\dots\dots\dots + d(x_m,x_{m+1})$$
$$\leq [L^{n-1} + L^{n-2} + L^{n-3} + L^{n-4} + \dots\dots\dots\dots + L^m] d(x_0,x_1)$$
$$\leq \frac{L^m}{1-L} d(x_0,x_1)$$

For a natural number N_1let $c < 0$ such that$\frac{L^m}{1-L} d(x_0,x_1) < c, \forall\, m \geq N_1$.

Thus $d(x_n,x_m) \leq \frac{L^m}{1-L} d(x_0,x_1) < c$for $n > m$. Therefore $\{x_n\}$ is a Cauchy sequence in a cone metric space (X,d), $\exists\, z^* \in X$ such that $x_n \to z^*$ as $n \to \infty$. As T is continuous, so $T \lim_{n\to\infty} x_n = Tz^*$ implies $\lim_{n\to\infty} Tx_n = Tz^*$ implies $\lim_{n\to\infty} x_{n-1} = Tz^*$ implies $Tz^* = z^*$. Hence z^* is a fixed point T.

For uniqueness of fixed point z^*, let $z^{**}(z^* \neq z^{**})$ be another fixed point of T.

Now consider $d(z^*,z^{**}) = d(Tz^*,Tz^{**})$

$$\geq \alpha_1 d(z^*,z^{**}) + \alpha_2[\{d(z^*,z^{**}) + d(Tz^*,Tz^{**})\}] + \alpha_3[\{d(z^*,Tz^*) + d(z^{**},Tz^{**})\}] + \alpha_4 \frac{[d(Tz^*,z^{**}) + d(z^*,z^{**})][1 + d(z^{**},Tz^{**})]}{1 + d(z^*,z^{**})} + \alpha_5 \frac{[d(Tz^*,z^{**}) + d(z^*,Tz^*)] d(z^{**},Tz^{**})}{d(z^*,z^{**})} + +\alpha_6 \frac{d(Tz^*,z^{**}) + d(Tz^*,Tz^{**}) - (Tz^*,z^{**})^2 d(Tz^*,Tz^{**}) - d(Tz^*,z^{**}) d(Tz^*,Tz^{**})^2}{1 - d(Tz^*,z^{**}) d(Tz^*,Tz^{**})}$$

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$$\geq \alpha_1 d(z^*,z^{**}) + \alpha_2[\{d(z^*,z^{**}) + d(z^*,z^{**})\}] + \alpha_3[\{d(z^*,z^*) + d(z^{**},z^{**})\}] + \alpha_4 \frac{[d(z^*,z^{**}) + d(z^*,z^{**})][1 + d(z^{**},z^{**})]}{1 + d(z^*,z^{**})} + \alpha_5 \frac{[d(z^*,z^{**}) + d(z^*,z^*)]d(z^{**},z^{**})}{d(z^*,z^{**})} + \alpha_6 \frac{d(z^*,z^{**}) + d(z^*,z^{**}) - d(z^*,z^{**})^2 d(z^*,z^{**}) - d(z^*,z^{**})d(z^*,z^{**})^2}{1 - d(z^*,z^{**})d(z^*,z^{**})}$$

$$= \alpha_1 d(z^*,z^{**}) + \alpha_2[\{d(z^*,z^{**}) + d(z^*,z^{**})\}] + \alpha_4[d(z^*,z^{**}) + d(z^*,z^{**})] + \alpha_6[d(z^*,z^{**}) + d(z^*,z^{**})]$$

$$d(z^*,z^{**}) \geq +\alpha_1 d(z^*,z^{**}) + 2(\alpha_2 + \alpha_4 + \alpha_6)d(z^*,z^{**})$$

$$d(z^*,z^{**}) \geq [\alpha_1 + 2(\alpha_2 + \alpha_4 + \alpha_6)]d(z^*,z^{**})$$

Which is contradiction thus $d(z^*,z^{**}) = 0$

Similarly, we can show that $d(z^{**},z^*) = 0$ which implies $z^* = z^{**}$ is the unique fixed point of T.

Remark 3.2. If we put $\alpha_2 = \alpha_3 = \alpha_5 = \alpha_6 = 0$ in theorem (3.1) we will get result of [22].

Example 3.3. Let $X = [0,1]$ with a complete cone metric defined by

$d(x,y) = |x|$ for all $x, y \in X$,

and define the continuous self-mapping T by $Tx = \frac{x}{2}$ with $\alpha = \frac{1}{8}$, $\beta = \frac{1}{10}$, $\gamma = \frac{1}{12}$. Then T satisfies all the conditions of Theorem 3.1, and $x = 0$ is the unique fixed point of T in X.

Theorem 3.4. Let (X,d) be a complete cone metric space and let $T: X \to X$ be amapping satisfying the following condition.

$$d(Tx,Ty) \leq q\, max \left\{ \frac{d(x,y) + d(x,Ty)}{2}, \frac{d(x,Tx) + d(y,Ty)}{2}, \frac{d(x,y)\left[d(x,Tx) + \sqrt{d(Tx,Ty)d(y,Ty)}\right]^2}{[d(Tx,Ty) + d(x,y)]^2} \right\} \quad (3.2)$$

Proof: For $x_0 \in X$ we define a sequence $\{x_n\}$ by $x_1 = Tx_0, x_2 = Tx_1, \dots\dots\dots\dots, x_{n+1} = Tx_n$.

Consider $d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n)$

$$d(x_n,x_{n+1}) \leq q\, max \left\{ \frac{d(x_{n-1},x_n) + d(x_{n-1},Tx_n)}{2}, \frac{d(x_{n-1},Tx_{n-1}) + d(x_n,Tx_n)}{2}, \frac{d(x_{n-1},x_n)\left[d(x_{n-1},Tx_{n-1}) + \sqrt{d(Tx_{n-1},Tx_n)d(x_n,Tx_n)}\right]^2}{[d(Tx_{n-1},Tx_n) + d(x_{n-1},x_n)]^2} \right\}$$

$$= q\, max \left\{ \frac{d(x_{n-1},x_n) + d(x_{n-1},x_{n+1})}{2}, \frac{d(x_{n-1},x_n) + d(x_n,x_{n+1})}{2}, \frac{d(x_{n-1},x_n)\left[d(x_{n-1},x_n) + \sqrt{d(x_n,x_{n+1})d(x_n,x_{n+1})}\right]^2}{[d(x_n,x_{n+1}) + d(x_{n-1},x_n)]^2} \right\}$$

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$$d(x_n,x_{n+1}) \leq q\ max\left\{\frac{d(x_n,x_{n+1})}{2},\frac{d(x_{n-1},x_{n+1})}{2}, d(x_{n-1},x_n)\right\}$$

$$d(x_n,x_{n+1}) \leq q\ d(x_{n-1},x_n)$$

In the similar fashion, we can find

$$d(x_n,x_{n+1}) \leq q^n d(x_{n-1},x_n)$$

For $n > m$ we have

$$d(x_n,x_m) \leq d(x_{n-1},x_n) + d(x_{n-2},x_{n-1}) + \dots\dots\dots\dots\dots + d(x_m,x_{m+1})$$

$$\leq [q^{n-1} + q^{n-2} + q^{n-3} + q^{n-4} + \dots\dots\dots\dots + q^m]d(x_0,x_1)$$

$$\leq \frac{q^m}{1-q} d(x_0,x_1)$$

For a natural number N_1let $c < 0$ such that$\frac{q^m}{1-q}d(x_0,x_1) < c, \forall\, m \geq N_1$.

Thus $d(x_n,x_m) \leq \frac{q^m}{1-q}d(x_0,x_1) < c$for $n > m$. Therefore $\{x_n\}$ is a Cauchy sequence in a complete cone metric space (X,d), $\exists\, z^* \in X$ such that $x_n \to z^*$ as $n \to \infty$. i.e., $\lim_{n\to\infty} x_n \to z^*$ As T is continuous, so $T\lim_{n\to\infty} x_n = Tz^*$ implies $\lim_{n\to\infty} Tx_n = Tz^*$ implies $\lim_{n\to\infty} x_{n+1} = Tz^*$ implies $Tz^* = z^*$. Hence z^* is a fixed point T.

For uniqueness of fixed point z^*, let $z^{**}(z^* \neq z^{**})$ be another fixed point of T.

consider $d(z^*,z^{**}) = d(Tz^*,Tz^{**})$

$$\leq q\ max\left\{\frac{d(z^*,z^{**}) + d(z^*,Tz^{**})}{2},\frac{d(z^*,Tz^*) + d(z^{**},Tz^{**})}{2}, \frac{d(z^*,z^{**})\left[d(z^*,Tz^*) + \sqrt{d(Tz^*,Tz^{**})d(z^{**},Tz^{**})}\right]^2}{[d(Tz^*,Tz^{**}) + d(z^*,z^{**})]^2}\right\}$$

$$d(z^*,z^{**}) \leq q\ max\left\{\frac{d(z^*,z^{**}) + d(z^*,z^{**})}{2},\frac{d(z^*,z^*) + d(z^{**},z^{**})}{2}, \frac{d(z^*,z^{**})\left[d(z^*,z^*) + \sqrt{d(z^*,z^{**})d(z^{**},z^{**})}\right]^2}{[d(z^*,z^{**}) + d(z^*,z^{**})]^2}\right\}$$

$$d(z^*,z^{**}) \leq q d(z^*,z^{**})$$

This implies $z^* = z^{**}$ is a unique fixed point of T.

Example3. 5. Let (X,d) be the complete cone metric space with$X = \mathbb{R}$, defined by $d(x,y) = |x|$ for all $x,y \in X$ with $Tx = \frac{x}{8}\ \forall x \in X$. Then

$$d(Tx,Ty) = \left|\frac{x}{8}\right| \leq \left|\frac{x}{4}\right| = \frac{1}{4}|x| = q(x,y).$$

Thus for $q = \frac{1}{4}$ satisfies the conditions pf the theorem 3.2 having $x = 0$ as the unique fixed point of T.

Remark: The existence and uniqueness of fixed point can be proved for two mappings (i.e., for $T$and S) in a similar way as in the above theorem.

Theorem $\mathbf{3.6}$.Let (X, d)be a complete cone metric space and let $T: X \to X$be amapping satisfying the following condition.

$$d(Tx, Ty) \le a\varphi d(x, y) + b\varphi \max\{d(x, Tx) + d(x, y)\} + c\varphi \left\{ \frac{d(x, y)\left[1 + \sqrt{d(x, y) d(x, Tx)}\right]^2}{[1 + d(x, y)]^2} \right\} \tag{3.3}$$

for all $x, y \in X$, $a, b, c \ge 0$ with $a + b + c < 1$ and φ is a comparison function. Then Thas a unique fixed point.

Proof: Let$x_0 \in X$ be arbitrary point and $\{x_n\}$be a sequence in X, defined as follows

$x_{n+1} = Tx_n, n = 0,1,2,3,4, \dots \dots \dots$

Consider

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$
$$\le a\varphi d(x_{n-1}, x_n) + b\varphi \max\{d(x_{n-1}, Tx_{n-1}) + d(x_{n-1}, x_n)\} + c\varphi \left\{ \frac{d(x_{n-1}, x_n)\left[1 + \sqrt{d(x_{n-1}, x_n) d(x_{n-1}, Tx_{n-1})}\right]^2}{[1 + d(x_{n-1}, x_n)]^2} \right\}$$

$$= a\varphi d(x_{n-1}, x_n) + b\varphi \max\{d(x_{n-1}, x_n) + d(x_{n-1}, x_n)\} + c\varphi \left\{ \frac{d(x_{n-1}, x_n)\left[1 + \sqrt{d(x_{n-1}, x_n) d(x_{n-1}, x_n)}\right]^2}{[1 + d(x_{n-1}, x_n)]^2} \right\}$$

$$= (a + b + c)\varphi d(x_{n-1}, x_n)$$

Since $\varphi(t) \le t \ \forall t \ge 0$,

$$\Rightarrow d(x_n, x_{n+1}) = (a + b + c) d(x_{n-1}, x_n)$$
$$\Rightarrow d(x_n, x_{n+1}) \le L d(x_{n-1}, x_n)$$

Where $L = (a + b + c) < 1$

Continuing in the similar fashion we have

$$d(x_n, x_{n+1}) \le L^n d(x_0, x_1)$$

Taking limit $n \to \infty, L^n \to 0$. Therefore

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0$$

Which prove that $\{x_n\}$ is a Cauchy sequence in complete cone metric space X. So there exists $z^* \in X$ such that

$$\lim_{n \to \infty} x_n = z^*.$$

Also since T is continuous function so we have

$$Tz^* = T \lim_{n \to \infty} x_n = \lim_{n \to \infty} Tx_n + \lim_{n \to \infty} x_{n+1} = z^*$$

Therefore z^* is the fixed point of T.

Uniqueness. Let $z^* \neq z^{**}$ are two distinct fixed points of T then consider

$d(z^*, z^{**}) = d(Tz^*, Tz^{**})$.

$$\leq a\varphi d(z^*, z^{**}) + b\varphi \max\{d(z^*, Tz^*) + d(z^*, z^{**})\} + c\varphi \left\{ \frac{d(z^*, z^{**})\left[1 + \sqrt{d(z^*, z^{**}) d(z^*, Tz^*)}\right]^2}{[1 + d(z^*, z^{**})]^2} \right\}$$

$$d(z^*, z^{**}) \leq a\varphi d(z^*, z^{**}) + b\varphi\{d(z^*, z^*) + d(z^*, z^{**})\} + c\varphi \left\{ \frac{d(z^*, z^{**})}{[1 + d(z^*, z^{**})]^2} \right\}$$

$$d(z^*, z^{**}) \leq (a + b + c)\varphi d(z^*, z^{**})$$

Since $\varphi(t) \leq t \; \forall t \geq 0$,

$$\Rightarrow d(z^*, z^{**}) \leq (a + b + c)\varphi d(z^*, z^{**})$$

For $a + b + c < 1$ the above inequality is possible only if $d(z^*, z^{**}) = 0$. Similarly we can show that $d(z^{**}, z^*) = 0$, which implies that $z^* = z^{**}$. Hence the fixed point of T is unique.

Example 3.7. Let (X, d) be the complete cone metric space with $X = \mathbb{R}$, defined by $d(x, y) = |x|$ for all $x, y \in X$ with $Tx = \frac{x}{16} \; \forall x \in X$. Then

$$d(Tx, Ty) = \left|\frac{x}{16}\right| \leq \left|\frac{x}{8}\right| = \frac{1}{8}|x| = \frac{1}{4}\frac{1}{2}|x| = a\varphi(x, y).$$

Thus for $a = \frac{1}{4}$ and $\varphi(t) = \frac{1}{2}$ for all $t \geq 0$ satisfies the conditions of the theorem 3.3 having $x = 0$ as the unique fixed point of T.

4 Conclusion

In this paper, we have proved some new fixed point results for single self-mapping using expanding and comparison functions satisfying contraction conditions in dislocated quasi metric spaces. These established results improve and modify results due to Mohammad *et, al.* [18], [19]. Rahman and Sarwar [22].

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