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New existence and uniqueness results for a nonlinear fractional order boundary value problem

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ABSTRACT

In this article, we consider a nonlinear fractional order threepoint boundary value problem. Some new existence and uniqueness results are obtained by using Kannan's fixed point theorem and one another fixed point theorem by Reich.

Keywords :*Banach space, Boundary Value Problem, Caputo Derivative,Fixed Point Theorem, Riemann Liouville Integral.*

I.INTRODUCTION

Fractional differential equations are being used in various fields of science and engineering such as control system, electrochemistry, electromagnetics, viscoelasticity, physics, biophysics, porous media, blood flow phenomena, electrical circuits, biology, fitting of experimental data etc. [1-3]. Due to these features, models of fractional order become more practical and realistic than the models of integer-order. Existence of solutions to differential equations of fractional order have received considerable interest in recent years. There are several papers dealing with the existence and uniqueness of solution to initial and boundary value problem of differential equation of fractional order. For some recent development on the topic, see [4-15] and the references therein.

This paper deals with the existence and uniqueness of solutions for the following three-point fractional integral boundary value problem:

$$\begin{cases} D^k z(\xi) = h\big(\xi, z(\xi)\big), \qquad \xi \in [0,1] \\ z(\tau) = z'(0) = z''(0) = \cdots = z^{n-2}(0) = 0, I^\alpha z(1) = 0, \qquad 0 < \tau < 1 \end{cases} \tag{1.1}$$

where $n-1 < k \leq n, n \in \mathbb{N}, n \geq 3$ and D^kdenotes the caputo fractional derivative of order $k, I^\alpha$is the Riemann-Liouville fractional integral of order $\alpha, h: [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and

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$\tau^{n-1} \neq \frac{\Gamma(n)}{(\alpha+n-1)(\alpha+n-2)\dots(\alpha+1)}$. By $C([0,1],\mathbb{R})$, we denote the Banach space of all continuous functions from [0,1] into $\mathbb{R}$ with the norm

$\|z\| = \sup\{|z(\xi)|: \xi \in [0,1]\}$.

II. PRELIMINARIES

Definition 2.1. For a continuous function $h: [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order k is defined as:

$$D^k h(\xi) = \frac{1}{\Gamma(n-k)} \int_0^{\xi} (\xi - s)^{n-k-1} h^{(n)}(s) ds, \qquad n = [k] + 1$$

provided that $h^{(n)}(\xi)$ exists, where $[k]$ denotes the integer part of the real number k.

Definition 2.2. The Riemann-Liouville fractional integral of order α for a continuous function $h(\xi)$ is defined as:

$$I^{\alpha} h(\xi) = \frac{1}{\Gamma(\alpha)} \int_0^{\xi} (\xi - s)^{\alpha-1} h(s) ds, \qquad \alpha > 0$$

provided that such integral exists.

Lemma 2.3. ([16]) Let $k > 0$, then

$I^k D^k z(\xi) = z(\xi) + c_0 + c_1\xi + c_2\xi^2 + \cdots + c_{n-1}\xi^{n-1}$,

for some $c_i \in \mathbb{R}, i = 0,1,2,\dots,n-1$, where n is the smallest integer greater than or equal to k.

Lemma 2.4. Let $\tau^{n-1} \neq \frac{\Gamma(n)}{(\alpha+n-1)(\alpha+n-2)\dots(\alpha+1)}$, $n-1 < k \leq n, 0 < \tau < 1$. Then for $w \in C([0,1],\mathbb{R})$, the problem

$$\begin{cases} D^k z(\xi) = w(\xi), \qquad \xi \in [0,1] \\ z(\tau) = z'(0) = z''(0) = \cdots = z^{n-2}(0) = 0, I^{\alpha} z(1) = 0, \qquad 0 < \tau < 1 \end{cases} \tag{2.1}$$

has a unique solution

$$\begin{aligned} z(\xi) &= \frac{1}{\Gamma(k)} \int_0^{\xi} (\xi - s)^{k-1} w(s) ds - \frac{1}{\Gamma(k)} \int_0^{\tau} (\tau - s)^{k-1} w(s) ds \\ &+ \frac{(\tau^{n-1} - \xi^{n-1})\Gamma(\alpha+n)}{\Gamma(\alpha+k)[\Gamma(n) - \tau^{n-1}(\alpha+n-1)(\alpha+n-2)\dots(\alpha+1)]} \int_0^1 (1-s)^{\alpha+k-1} w(s) ds \\ &- \frac{(\tau^{n-1} - \xi^{n-1})\Gamma(\alpha+n)}{\Gamma(\alpha+1)\Gamma(k)[\Gamma(n) - \tau^{n-1}(\alpha+n-1)(\alpha+n-2)\dots(\alpha+1)]} \int_0^{\tau} (\tau - s)^{k-1} w(s) ds \end{aligned} \tag{2.2}$$

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Proof. By applying Lemma 2.3, we may reduce (2.1) to an equivalent integral equation

$$z(\xi) = \frac{1}{\Gamma(k)}\int_0^{\xi}(\xi - s)^{k-1}w(s)ds - c_0 - c_1\xi + c_2\xi^2 - \cdots - c_{n-1}\xi^{n-1} \tag{2.3}$$

for some $c_i \in \mathbb{R}, i = 0,1,2, \dots, n-1.$

From $z'(0) = 0$, it follows $c_1 = 0$.

Also $z''(0) = 0 \Rightarrow c_2 = 0$

Continuing in this way we have $z^{n-2}(0) = 0 \Rightarrow c_{n-2} = 0$

Thus (2.3) becomes

$$z(\xi) = \frac{1}{\Gamma(k)}\int_0^{\xi}(\xi - s)^{k-1}w(s)ds - c_0 - c_{n-1}\xi^{n-1} \tag{2.4}$$

$$z(\tau) = 0 \Rightarrow \frac{1}{\Gamma(k)}\int_0^{\tau}(\tau - s)^{k-1}w(s)\,ds - c_0 - c_{n-1}\tau^{n-1} = 0 \tag{2.5}$$

Now

$$I^{\alpha}[z(\xi)] = \frac{1}{\Gamma(\alpha)}\int_0^{\xi}(\xi - s)^{\alpha-1}[I^k w(s) - c_0 - c_{n-1}s^{n-1}]ds$$

$$= I^{\alpha}I^k w(\xi) - \frac{c_0\xi^{\alpha}}{\Gamma(\alpha+1)} - \frac{c_{n-1}\xi^{\alpha+n-1}\Gamma(n)}{\Gamma(\alpha+n)}$$

$$= \frac{1}{\Gamma(\alpha+k)}\int_0^{\xi}(\xi - s)^{\alpha+k-1}w(s)ds - \frac{c_0\xi^{\alpha}}{\Gamma(\alpha+1)} - \frac{c_{n-1}\xi^{\alpha+n-1}\Gamma(n)}{\Gamma(\alpha+n)}$$

$$I^{\alpha}[z(1)] = 0 \Rightarrow \frac{1}{\Gamma(\alpha+k)}\int_0^{1}(1 - s)^{\alpha+k-1}w(s)\,ds - \frac{c_0}{\Gamma(\alpha+1)} - \frac{c_{n-1}\Gamma(n)}{\Gamma(\alpha+n)} = 0 \tag{2.6}$$

solving (2.5) and (2.6) for c_0, c_{n-1}, we have:

$$c_{n-1} = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha+k)[\Gamma(n) - \tau^{n-1}(\alpha+n-1)(\alpha+n-2)\dots(\alpha+1)]}\int_0^1(1-s)^{\alpha+k-1}w(s)ds$$
$$- \frac{\Gamma(\alpha+n)}{\Gamma(\alpha+1)\Gamma(k)[\Gamma(n) - \tau^{n-1}(\alpha+n-1)(\alpha+n-2)\dots(\alpha+1)]}\int_0^{\tau}(\tau - s)^{k-1}w(s)ds$$

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and

$$c_0 = \frac{1}{\Gamma(k)}\int_0^\tau (\tau-s)^{k-1} w(s)ds$$
$$-\frac{\tau^{n-1}\Gamma(\alpha+n)}{\Gamma(\alpha+k)[\Gamma(n)-\tau^{n-1}(\alpha+n-1)(\alpha+n-2)\dots(\alpha+1)]}\int_0^1 (1-s)^{\alpha+k-1} w(s)ds$$
$$+\frac{\tau^{n-1}\Gamma(\alpha+n)}{\Gamma(\alpha+1)\Gamma(k)[\Gamma(n)-\tau^{n-1}(\alpha+n-1)(\alpha+n-2)\dots(\alpha+1)]}\int_0^\tau (\tau-s)^{k-1} w(s)ds$$

putting the values of $c_i$in (2.3), we obtain the solution (2.2). ∎

Transform the BVP (1.1) into a fixed point problem. In view of Lemma 2.4, we consider the operator $T: C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ defined by

$$T(z)(\xi) = \frac{1}{\Gamma(k)}\int_0^\xi (\xi-s)^{k-1} h(s,z(s))ds - \frac{1}{\Gamma(k)}\int_0^\tau (\tau-s)^{k-1} h(s,z(s))ds$$
$$+\frac{(\tau^{n-1}-\xi^{n-1})\Gamma(\alpha+n)}{\Gamma(\alpha+k)[\Gamma(n)-\tau^{n-1}(\alpha+n-1)(\alpha+n-2)\dots(\alpha+1)]}$$
$$\times\int_0^1 (1-s)^{\alpha+k-1} h(s,z(s))ds$$
$$-\frac{(\tau^{n-1}-\xi^{n-1})\Gamma(\alpha+n)}{\Gamma(\alpha+1)\Gamma(k)[\Gamma(n)-\tau^{n-1}(\alpha+n-1)(\alpha+n-2)\dots(\alpha+1)]}$$
$$\times\int_0^\tau (\tau-s)^{k-1} h\big(s,z(s)\big)ds$$

For the sake of our convenience, let us set

$$\Omega = \frac{2}{\Gamma(k+1)}$$
$$+\frac{\Gamma(\alpha+n)}{\Gamma(\alpha+1)\Gamma(k+1)|\Gamma(n)-\tau^{n-1}(\alpha+n-1)(\alpha+n-2)\dots(\alpha+1)|} \quad (2.7)$$

$$+\frac{\Gamma(\alpha+n)}{\Gamma(\alpha+k+1)|\Gamma(n)-\tau^{n-1}(\alpha+n-1)(\alpha+n-2)\dots(\alpha+1)|} \quad (2.8)$$

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Theorem 2.5. Kannan's fixed point theorem Let(X, d) be a complete metric space. If there exists $\alpha \in [0, \frac{1}{2})$ such that that $T : X \to X$ satisfies

$$d(Tx, Ty) \leq \alpha[d(Tx, x) + d(Ty, y)] \ \ for\ all\ x, y \in X$$

Then T has a unique fixed point.

Theorem 2.6.([17]) Let (X, d) be a complete metric space and let a, b, c be non-negative numbers with $a + b + c < 1$. Further suppose that $T : X \to X$ satisfies

$$d(Tx, Ty) \leq ad(x, y) + bd(Tx, x) + cd(Ty, y)\ for\ all\ x, y \in X$$

Then T has a unique fixed point.

III. MAIN RESULTS

First result is based on Kannan's fixed point theorem.

Theorem 3.1. Assume that

(I) There exist $L > 0$ such that $|h(\xi, z_1) - h(\xi, z_2)| \leq L(|Tz_1 - z_1| + |Tz_2 - z_2|)$ for each $\xi \in [0,1]$, and all $z_1, z_2 \in \mathbb{R}$.

(II) $L\Omega < \frac{1}{2}$,whereΩ is defined by (2.8), then the BVP (1.1) has a unique solution on [0,1].

Proof. Obviously, the fixed points of the operator T defined by (2.7) are solution of the problem (1.1). We shall use Kannan's fixed point theorem to prove that T has a unique fixed point.

Let $z_1 , z_2 \in C([0,1, \mathbb{R})$, then for each $\xi \in [0,1]$, we have

$$|T(z_1)(\xi) - T(z_2)(\xi)| \leq \frac{1}{\Gamma(k)} \int_0^{\xi} (\xi - s)^{k-1} |h(s, z_1(s)) - h(s, z_2(s))| ds$$

$$+ \frac{1}{\Gamma(k)} \int_0^{\tau} (\tau - s)^{k-1} |h(s, z_1(s)) - h(s, z_2(s))| ds$$

$$+ \frac{\Gamma(\alpha + n)}{\Gamma(\alpha + k)|\Gamma(n) - \tau^{n-1}(\alpha + n - 1)(\alpha + n - 2) \ldots (\alpha + 1)|} \times \int_0^1 (1 - s)^{\alpha+k-1} |h(s, z_1(s)) - h(s, z_2(s))| ds$$

$$+ \frac{\Gamma(\alpha + n)}{\Gamma(\alpha + 1)\Gamma(k)|\Gamma(n) - \tau^{n-1}(\alpha + n - 1)(\alpha + n - 2) \ldots (\alpha + 1)|}$$

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$$\times \int_0^\tau (\tau - s)^{k-1} |h(s, z_1(s)) - h(s, z_2(s))| ds$$

$$\leq [L(\|Tz_1 - z_1)\| + \|Tz_2 - z_2\|)] \left[\frac{1}{\Gamma(k)} \int_0^\xi (\xi - s)^{k-1} ds + \frac{1}{\Gamma(k)} \int_0^\tau (\tau - s)^{k-1} ds\right.$$
$$+ \frac{\Gamma(\alpha + n)}{\Gamma(\alpha + 1)\Gamma(k) |\Gamma(n) - \tau^{n-1}(\alpha + n - 1)(\alpha + n - 2) \dots (\alpha + 1)|} \int_0^\tau (\tau - s)^{k-1} ds$$
$$\left. + \frac{\Gamma(\alpha + n)}{\Gamma(\alpha + k) |\Gamma(n) - \tau^{n-1}(\alpha + n - 1)(\alpha + n - 2) \dots (\alpha + 1)|} \int_0^1 (1 - s)^{\alpha + k - 1} ds\right]$$

$$\leq [L(\|Tz_1 - z_1)\| + \|Tz_2 - z_2\|)] \left[\frac{1}{\Gamma(k + 1)} + \frac{1}{\Gamma(k + 1)}\right.$$
$$+ \frac{\Gamma(\alpha + n)}{\Gamma(\alpha + 1)\Gamma(k + 1) |\Gamma(n) - \tau^{n-1}(\alpha + n - 1)(\alpha + n - 2) \dots (\alpha + 1)|}$$
$$\left. + \frac{\Gamma(\alpha + n)}{\Gamma(\alpha + k + 1) |\Gamma(n) - \tau^{n-1}(\alpha + n - 1)(\alpha + n - 2) \dots (\alpha + 1)|}\right]$$

Thus

$\|Tz_1 - Tz_2\| \leq L\Omega \|Tz_1 - z_1\| + L\Omega \|Tz_2 - z_2\|$

As $\Omega < \frac{1}{2}$, therefore, all the conditions of Kannan's fixed point theorem are satisfied. Hence, T has a unique fixed point which is a solution of the problem (1.1). ∎

Theorem 3.2. Assume that

(III) there exist non-negative constants L_1, L_2, L_3 such that $|h(\xi, z_1) - h(\xi, z_2)| \leq L_1|z_1 - z_2| + L_2|Tz_1 - z_1| + L_3|Tz_2 - z_2|$ for each $\xi \in [0,1]$, and all $z_1, z_2 \in \mathbb{R}$.

(IV) $L_1 + L_2 + L_3 < \frac{1}{\Omega}$, where Ω is defined by (2.8), then the BVP (1.1) has a unique solution on [0,1].

Proof. We shall use Theorem 2.6 to prove that T has a unique fixed point.

Let $z_1, z_2 \in C([0,1], \mathbb{R})$, then for each $\xi \in [0,1]$, we have

$$|T(z_1)(\xi) - T(z_2)(\xi)| \leq \frac{1}{\Gamma(k)} \int_0^\xi (\xi - s)^{k-1} |h(s, z_1(s)) - h(s, z_2(s))| ds$$

$$+ \frac{1}{\Gamma(k)} \int_0^\tau (\tau - s)^{k-1} |h(s, z_1(s)) - h(s, z_2(s))| ds$$

$$+ \frac{\Gamma(\alpha + n)}{\Gamma(\alpha + k) |\Gamma(n) - \tau^{n-1}(\alpha + n - 1)(\alpha + n - 2) \dots (\alpha + 1)|}$$
$$\times \int_0^1 (1 - s)^{\alpha + k - 1} |h(s, z_1(s)) - h(s, z_2(s))| ds$$

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$$+\frac{\Gamma(\alpha+n)}{\Gamma(\alpha+1)\Gamma(k)\left|\Gamma(n)-\tau^{n-1}(\alpha+n-1)(\alpha+n-2)\dots(\alpha+1)\right|}$$
$$\times\int_0^\tau(\tau-s)^{k-1}\left|h\big(s,z_1(s)\big)-h\big(s,z_2(s)\big)\right|ds$$
$$\le\left[L_1\|z_1-z_2\|+L_2\|Tz_1-z_1\|+L_3\|Tz_2-z_2\|\right]\left[\frac{1}{\Gamma(k)}\int_0^\xi(\xi-s)^{k-1}ds+\frac{1}{\Gamma(k)}\int_0^\tau(\tau-s)^{k-1}ds\right.$$
$$+\frac{\Gamma(\alpha+n)}{\Gamma(\alpha+1)\Gamma(k)\left|\Gamma(n)-\tau^{n-1}(\alpha+n-1)(\alpha+n-2)\dots(\alpha+1)\right|}\int_0^\tau(\tau-s)^{k-1}ds$$
$$\left.+\frac{\Gamma(\alpha+n)}{\Gamma(\alpha+k)\left|\Gamma(n)-\tau^{n-1}(\alpha+n-1)(\alpha+n-2)\dots(\alpha+1)\right|}\int_0^1(1-s)^{\alpha+k-1}ds\right]$$
$$\le\left[L_1\|z_1-z_2\|+L_2\|Tz_1-z_1\|+L_3\|Tz_2-z_2\|\right]\left[\frac{1}{\Gamma(k+1)}+\frac{1}{\Gamma(k+1)}\right.$$
$$+\frac{\Gamma(\alpha+n)}{\Gamma(\alpha+1)\Gamma(k+1)\left|\Gamma(n)-\tau^{n-1}(\alpha+n-1)(\alpha+n-2)\dots(\alpha+1)\right|}$$
$$\left.+\frac{\Gamma(\alpha+n)}{\Gamma(\alpha+k+1)\left|\Gamma(n)-\tau^{n-1}(\alpha+n-1)(\alpha+n-2)\dots(\alpha+1)\right|}\right]$$

Thus

$$\|Tz_1-Tz_2\|\le L_1\Omega\|z_1-z_2\|+L_2\Omega\|Tz_1-z_1\|+L_3\Omega\|Tz_2-z_2\|$$

As$L_1+L_2+L_3<\frac{1}{\Omega}$, therefore, all the conditions of Theorem 2.6 are satisfied. Hence, T has a unique fixed point which is a solution of the problem (1.1). ■

IV CONCLUSION

In this work we have presented a new type of three-point nonlinear fractionalboundary value problem of arbitrary order and find its existence and uniqueness of solutions using some new fixed point theorems which have not been used in the Literature.

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