ON THE EXISTENCE RESULTS FOR A NONLINEAR DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

Charu[1], Sachin Kumar Verma[2]

[1]*Department of Mathematics, R.K.S.D(P.G) CollegeKaithal, Haryana, India.*

[2]*Department of Mathematics, NIT Hamirpur, HP-177005, India.*

ABSRACT

In this paper, we studythe existence and uniqueness of solutions fora nonlinear differential equations of fractional order with three point boundary conditions.The main tools used are Krasnoselaskii's fixed point theorem and Banach contraction principle.

Keywords :Banach space, Boundary Value Problem, Caputo Derivative, Fixed Point Theorem, Riemann Liouville Integral.

I. INTRODUCTION

Nonlinear fractional differential equations have recently used to be a valuable tools in various fields of engineering and science such as porous media, biology, electrochemistry, fitting of experimental, physics,economics, control theory etc, see [1-4]. These fractional derivatives provide an excellent feature for the hereditary properties of different materials processes and description of memory, see [1].There has been a significant development in the theory of initial and boundary value problems for nonlinear fractional differential equations, see [5-12].

This article deals with the existence and uniqueness of solutions for the following three-point boundary value problem of fractional differential equation:

$$D_c^l y(\tau) = f(\tau, y(\tau)), \quad 0 < \tau < 1, \quad l \in (3,4] \quad \text{(A)}$$

$$y(0) = 0, \; y'(0) = 0, \; y''(0) = 0, \quad \beta[I^k y](\xi) = y(1), \; 0 < \xi < 1 \quad \text{(B)}$$

ISBN : 978-93-87793-43-9

where D_c^l denotes the Caputo fractional derivative of order l, I^k is the Riemann-Liouville fractional integral of order$k$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm

$$\|y\| = sup\{|y(\tau)| : \tau \in [0,1]\}$$

II. PRELIMINARIES

Definition 2.1. For a continuous function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order is defined as

$$D_c^l g(\tau) = \frac{1}{\Gamma(m-l)} \int_0^\tau (\tau-u)^{(m-l-1)} g^{(m)}(u)\, du, \quad m-1 < l < m, \qquad m = [l]+1$$

provided that$g^{(m)}(\tau)$ exists, where [l] denotes the integral part of the real number l.

Definition2.2. The Riemann-Liouvillefractional integral of order $l$for a continuous function $g(\tau)$ is defined by

$$I^{(l)} g(\tau) = \frac{1}{\Gamma(l)} \int_0^\tau (\tau-u)^{(l-1)} g(u) du, \qquad l > 0$$

provided that such integral exists.

Definition 2.3.The Riemann-Liouville fractional derivative of order $l$for a continuous function $g(\tau)$ is defined by

$$D^{(l)} g(\tau) = \frac{1}{\Gamma(m-l)} \frac{d}{d\tau}^m \int_0^\tau (\tau-u)^{(m-l-1)} g(u) du, \quad m = [l]+1$$

provided that the right hand side is pointwise defined on $[0,\infty)$.

Lemma 2.1.[3]Let $l > 0$ then the fractional differential equation $D_c^l h(\tau) = 0$has a solution

$$h(\tau) = b_0 + b_1\tau + b_2\tau^2 + .. + b_{m-1}\tau^{m-1}, \qquad b_i \in \mathbb{R}, \qquad i = 0,1,2,\dots,m-1$$

where$m$ is the smallest integer greater than or equal to l.

Lemma 2.2.[3] Let $l > 0$ then

$$I^l D_c^l h(\tau) = h(\tau) + b_0 + b_1\tau + b_2\tau^2 + .. + b_{m-1}\tau^{m-1}, \qquad b_i \in \mathbb{R}, \qquad i = 0,1,2,\dots,m-1$$

where$m$ is the smallest integer or equal to l.

ISBN : 978-93-87793-43-9

Lemma 2.3. Let $\beta \neq \frac{\Gamma(k+3)}{6\xi^{k+2}}$, $3 < q \leq 4, 0 < \xi < 1.$

then for $v \in C([0,1],\mathbb{R})$ the problem

$$D_c^l y(\tau) = v(\tau) \quad 0 < \tau < 1, \tag{1}$$

$$y(0) = 0,\ y'(0) = 0,\ y''(0) = 0, \quad \beta[I^k y](\xi) = y(1) \tag{2}$$

has a unique solution

$$y(\tau) = \frac{1}{\Gamma(l)}\int_0^\tau (\tau-u)^{(l-1)} v(u)\,du - \frac{\Gamma(k+3)\tau^3}{\Gamma(l)\,[\Gamma(k+3) - 6\beta\xi^{k+2}]}\int_0^1 (1-u)^{l-1} v(u)du$$

$$+\frac{\beta k(k+1)(k+2)\tau^3}{\Gamma(l)\,[\Gamma(k+3) - 6\beta\xi^{k+2}]}\int_0^\xi \int_0^u (\xi-u)^{(k-1)} (u-r)^{(l-1)} v(r)\,dr du \tag{3}$$

Proof. By applying Lemma 2.2, we reduce (1) to an equivalent integral equation

$$y(\tau) = \frac{1}{\Gamma(l)}\int_0^\tau (\tau-u)^{(l-1)} v(u)\,du - c_0 - c_1\tau - c_2\tau^2 - c_3\tau^3 \tag{4}$$

for some $c_0, c_1, c_2, c_3 \in \mathbb{R}$

from$y(0) = y'(0) = y''(0) = 0$ it implies that $c_0 = c_1 = c_2 = 0$

put the value of c_0, c_1, c_2 in the equation (4)

$$y(\tau) = \frac{1}{\Gamma(l)}\int_0^\tau (\tau-u)^{(l-1)} v(u)\,du - c_3\tau^3 \tag{5}$$

Now using the Riemann-Liouville integral of order k for (4), we get

$$I^{(k)}y(\tau) = \frac{1}{\Gamma(k)}\int_0^\tau (\tau-u)^{(k-1)} [\frac{1}{\Gamma(l)}\int_0^u (u-r)^{(l-1)} v(r)\,dr - c_3u^3]\,du$$

$$= \frac{1}{\Gamma(k)}\frac{1}{\Gamma(l)}\int_0^\tau \int_0^u (\tau-u)^{(k-1)} (u-r)^{(l-1)} v(r)\,dr\,du - \frac{6c_3\tau^{k+2}}{\Gamma(k+3)}$$

since$\beta[I^k y](\xi) = y(1)$,we get

ISBN : 978-93-87793-43-9

$$\frac{\beta}{\Gamma(k)\Gamma(l)}\int_0^{\xi}\int_0^{u}(\xi-u)^{(k-1)}(u-r)^{(l-1)}v(r)\,dr\,du-\frac{6c_3\beta\tau^{k+2}}{\Gamma(k+3)}$$

$$=\frac{1}{\Gamma(l)}\int_0^1(1-u)^{l-1}v(u)du-c_3$$

Thus

$$c_3=\frac{\Gamma(k+3)\tau^3}{\Gamma(l)[\Gamma(k+3)-6\beta\xi^{k+2}]}\int_0^1(1-u)^{l-1}v(u)du-\frac{\beta k(k+1)(k+2)\tau^3}{\Gamma(l)[\Gamma(k+3)-6\beta\xi^{k+2}]}\int_0^{\xi}\int_0^{u}(\xi-u)^{(k-1)}(u-r)^{(l-1)}v(r)\,drdu$$

substituting the value of c_3 in (5),we get

$$y(\tau)=\frac{1}{\Gamma(l)}\int_0^{\tau}(\tau-u)^{(l-1)}v(u)\,du-\frac{\Gamma(k+3)\tau^3}{\Gamma(l)[\Gamma(k+3)-6\beta\xi^{k+2}]}\int_0^1(1-u)^{l-1}v(u)\,du$$

$$+\frac{\beta k(k+1)(k+2)\tau^3}{\Gamma(l)[\Gamma(k+3)-6\beta\xi^{k+2}]}\int_0^{\xi}\int_0^{u}(\xi-u)^{(k-1)}(u-r)^{(l-1)}v(r)\,drdu$$

For the sake of convenience, we set

$$\mathcal{E}=\frac{1}{\Gamma(l+1)}+\frac{\Gamma(k+3)}{\Gamma(l+1)[\Gamma(k+3)-6\beta\xi^{k+2}]}+\frac{\beta\Gamma(k+3)\xi^{k+l}}{\Gamma(k+l+1)[\Gamma(k+3)-6\beta\xi^{k+2}]}$$

III.MAIN RESULTS

Theorem 3.1 Suppose that there exists a constant$H>0$such that

(a)$|g(\tau,x_1)-g(\tau,x_2)|\le H|x_1-x_2|$for each$\tau\in[0,1]$and for all$x_1,x_2\in\mathbb{R}$

(b) $H\mathcal{E}<1$,where$\mathcal{E}$is defined as above then theB.V.P (A)-(B)has a unique solution on[0,1].

Proof. Transform the B.V.P.(A)-(B) into a fixed point problem. In view of Lemma2.3,we consider the operator

$G: C([0,1],\mathbb{R})\to C([0,1],\mathbb{R})$defined by

$$G(x)(\tau)=\frac{1}{\Gamma(l)}\int_0^{\tau}(\tau-u)^{(l-1)}g(u,x(u))du-\frac{\Gamma(k+3)\tau^3}{\Gamma(l)[\Gamma(k+3)-6\beta\xi^{k+2}]}\int_0^1(1-u)^{l-1}g(u,x(u))du$$

ISBN : 978-93-87793-43-9

$$+\frac{\beta k(k+1)(k+2)\tau^3}{\Gamma(l)[\Gamma(k+3)-6\beta\xi^{k+2}]}\int_0^{\xi}\int_0^{u}(\xi-u)^{(k-1)}(u-r)^{(l-1)}g(r,x(r))drdu$$

we use the Banach fixed theorem to prove that G has a fixed point. we will show that G is a contraction mapping.

Let$x_1, x_2 \in C([0,1],\mathbb{R})$, then for each $\tau \in [0,1]$, we have

$$|G(x_1,\tau)-G(x_2,\tau)|$$

$$\leq \frac{1}{\Gamma(l)}\int_0^{\tau}(\tau-u)^{(l-1)}|g(u,x_1(u))-g(u,x_2,(u))|du$$

$$+\frac{\Gamma(k+3)\tau^3}{\Gamma(l)[\Gamma(k+3)-6\beta\xi^{k+2}]}\int_0^1(1-u)^{l-1}|g(u,x_1(u))-g(u,x_2,(u))|du$$

$$+\frac{\beta k(k+1)(k+2)\tau^3}{\Gamma(l)[\Gamma(k+3)-6\beta\xi^{k+2}]}\int_0^{\xi}\int_0^{u}(\xi-u)^{(k-1)}(u-r)^{(l-1)}|g(r,x_1(r))-g(r,x_2,(r))|drdu$$

$$\leq \frac{H\|x_1-x_2\|}{\Gamma(l)}\int_0^{\tau}(\tau-u)^{(l-1)}du+\frac{H\|x_1-x_2\|\Gamma(k+3)}{\Gamma(l)[\Gamma(k+3)-6\beta\xi^{k+2}]}\int_0^1(1-u)^{l-1}du$$

$$+\frac{H\|x_1-x_2\|k(k+1)(k+2)\beta}{\Gamma(l)[\Gamma(k+3)-6\beta\xi^{k+2}]}\int_0^{\xi}\int_0^{u}(\xi-u)^{(k-1)}(u-r)^{(l-1)}drdu$$

$$=\frac{H\|x_1-x_2\|}{\Gamma(l+1)}+\frac{H\|x_1-x_2\|\Gamma(k+3)}{\Gamma(l+1)[\Gamma(k+3)-6\beta\xi^{k+2}]}$$

$$+\frac{H\|x_1-x_2\|k(k+1)(k+2)\beta\xi^{k+l}B(l+1;k)}{\Gamma(l+1)[\Gamma(k+3)-6\beta\xi^{k+2}]}$$

where$B$ is the beta function $B(l+1;k)=\frac{\Gamma(l+1)\Gamma(k)}{\Gamma(k+l+1)}$; we have

$$|G(x_1,\tau)-G(x_2,\tau)|\leq\frac{H\|x_1-x_2\|}{\Gamma(l+1)}+\frac{H\|x_1-x_2\|\Gamma(k+3)}{\Gamma(l+1)[\Gamma(k+3)-6\beta\xi^{k+2}]}$$

ISBN : 978-93-87793-43-9

$$+\frac{H\|x_1-x_2\|k(k+1)(k+2)\beta\xi^{k+l}}{\Gamma(k+l+1)[\Gamma(k+3)-6\beta\xi^{k+2}]}$$

thus

$$\|G(x_1)-G(x_2)\| \le H\mathcal{E}\|x_1-x_2\|$$

therefore G is a contraction mapping; Hence by Banach Fixed point theorem, we get that G has a fixed point which is solution of the problem (A)-(B). □

The next result is based on Krasnoselskii's fixed point theorem.

Theorem3.2 Let $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and the following hypotheses hold

(a) there exists a constant $H > 0$ such that

$|g(\tau,x_1) - g(\tau,x_2)| \le H|x_1-x_2|$ for each $\tau \in [0,1]$ and all $x_1,x_2 \in \mathbb{R}$

(b) $|g(\tau,x)| \le \Psi(\tau)$ *for all* $(\tau,x) \in [0,1] \times \mathbb{R}$ and $\Psi \in C([0,1] \times \mathbb{R}^+)$

(c) $H\left[\frac{1}{\Gamma(l+1)}+\frac{\Gamma(k+3)}{\Gamma(l+1)[\Gamma(k+3)-6\beta\xi^{k+2}]}+\frac{\beta\Gamma(k+3)\xi^{k+l}}{\Gamma(k+l+1)[\Gamma(k+3)-6\beta\xi^{k+2}]}\right] < 1$

then the boundary value problem (A)-(B) has atleast one solution on [0,1].

Proof. We suppose that $sup_{\tau\in[0,1]}|\Psi(\tau)| = \|\Psi\|$, we fix

$$\eta \ge \|\Psi\|\mathcal{E}$$

consider $B_\eta = \{y \in C[0,1], \mathbb{R} \mid \|y\| \le \eta\}$ we define the operators G_1 and G_2 on B_η as

$G_1(y)(\tau)$

$$= \frac{1}{\Gamma(l)}\int_0^\tau (\tau-u)^{(l-1)} g(u,y(u))du$$

$G_2(y)(\tau)$

$$= -\frac{\Gamma(k+3)\tau^3}{\Gamma(l)[\Gamma(k+3)-6\beta\xi^{k+2}]}\int_0^1 (1-u)^{l-1} g(u,y(u))du$$

$$+\frac{\beta k(k+1)(k+2)\tau^3}{\Gamma(l)[\Gamma(k+3)-6\beta\xi^{k+2}]}\int_0^\xi\int_0^u (\xi-u)^{(k-1)}(u-r)^{(l-1)} g(r,y(r))drdu$$

for $x_1,x_2 \in B_\eta$ we find that

ISBN : 978-93-87793-43-9

$$\|G_1x_1 + G_2x_2\| \leq \frac{\|\Psi\|}{\Gamma(l+1)} + \frac{\|\Psi\|\Gamma(k+3)}{\Gamma(l+1)[\Gamma(k+3) - 6\beta\xi^{k+2}]} + \frac{\|\Psi\|\beta\Gamma(k+3)\xi^{k+l}}{\Gamma(k+l+1)[\Gamma(k+3) - 6\beta\xi^{k+2}]}$$

$$= \|\Psi\|\mathcal{E} \leq \mathfrak{y}$$

Thus $G_1x_1 + G_2x_2 \in B_{\mathfrak{y}}$.it follows from *(a)* and *(c)*G_2 is a contraction mapping. From the continuity of g,we obtain that the operatorG_1 is continuous, also G_1 is uniformly bounded on $B_{\mathfrak{y}}$ as

$$\|G_1x_1\| \leq \frac{\|\Psi\|}{\Gamma(l+1)}$$

Now we will prove that the operator G_1 is compact.

Define $sup_{\tau\in[0,1]\times B_{\mathfrak{y}}} |g(\tau, x)| = \hat{g}$ then we have

$$|G_1(x)(\tau_1) - G_2(x)(\tau_2)| = \left|\frac{1}{\Gamma(l+1)}\int_0^{\tau_1}((\tau_1-u)^{(l-1)} - (\tau_2-u)^{(l-1)})g(u, x(u))du + \int_{\tau_1}^{\tau_2}(\tau_2-u)^{(l-1)}g(u, x(u))du\right|$$

$$\leq \frac{\hat{g}}{\Gamma(l+1)}|2((\tau_2-\tau_1)^l + {\tau_1}^l - {\tau_2}^l|$$

which is independent of x and approaches to zero as $\tau_1 \to \tau_2$. thus G_1 is equicontinuous hence by Arzela-Ascolli theorem,G_1 is compact on $B_{\mathfrak{y}}$.thus all the assumptions of Krasnoselaskii's fixed point theorem are satisfied. so the boundary value problem (A)-(B) has atleast one solution defined on [0,1]. □

IV.EXAMPLES

To illustrate our results, we consider two examples

Example 4.1 Consider the following three-point fractional integral boundary value problem

$$D_c^{\frac{7}{2}}x(\tau) = \frac{1}{(\tau+5)^2}\frac{|x|}{1+|x|}; \qquad \tau \in [0,1] \tag{4.1}$$

$x(0) = 0,\ x'(0) = 0,\ x''(0) = 0,\ \ \sqrt{3}[I^k x](\frac{1}{10}) = x(1)$; (4.2)

Set

$$\xi = \frac{1}{10}, l = \frac{7}{2}, k = \frac{5}{2}, m = 4, \beta = \sqrt{3} \neq \frac{\Gamma\left(\frac{11}{2}\right)}{(1/10)^{11/2}}$$

ISBN : 978-93-87793-43-9

$$g(\tau,x)=\frac{1}{(\tau+5)^2}\frac{|x|}{1+|x|}$$

Since $\|g(\tau,x)-g(\tau,y)\|\leq\frac{1}{25}\|x-y\|$

then*(a)* of theorem 3.1 is satisfied with $H=\frac{1}{25}$. Also we can easily show that

$$H\mathcal{E}=H\left[\frac{1}{\Gamma(l+1)}+\frac{\Gamma(k+3)}{\Gamma(l+1)[\Gamma(k+3)-6\beta\xi^{k+2}]}+\frac{\beta\Gamma(k+3)\xi^{k+l}}{\Gamma(k+l+1)[\Gamma(k+3)-6\beta\xi^{k+2}]}\right]$$

$$=\frac{1}{25}\left[\frac{1}{\Gamma\left(\frac{9}{2}\right)}+\frac{\Gamma\left(\frac{11}{2}\right)}{\Gamma\left(\frac{9}{2}\right)\left[\Gamma\left(\frac{11}{2}\right)-6\sqrt{3}(\frac{1}{10})^{\frac{7}{2}}\right]}+\frac{\Gamma\left(\frac{11}{2}\right)\sqrt{3}(\frac{1}{10})^{6}}{\Gamma(7)\left[\Gamma\left(\frac{11}{2}\right)-6\sqrt{3}(\frac{1}{10})^{\frac{7}{2}}\right]}\right]\approx 0.171<1$$

Hence all the conditions of theorem 3.1,the boundary value problem (4.1)-(4.2) has a unique solution on [0,1].

Example 4.2 Consider the following fractional differential equation

$$D_c^{\frac{9}{2}}x(\tau)=\frac{1}{(\tau+7)^2}\sin x\,,\tau\in[0,1] \quad (4.3)$$

with three-point boundary value conditions

$$x(0)=0,\quad x'(0)=0,\quad x''(0)=0,\quad \sqrt{2}\left[I^{\frac{3}{2}}x\right]\left(\frac{1}{4}\right)=x(1); \quad (4.4)$$

Set

$$l=\frac{9}{2},m=4,k=\frac{3}{2},\beta=\sqrt{2}\neq\frac{\Gamma\left(\frac{9}{2}\right)}{(1/4)^{9/2}}\quad and\quad g(\tau,x)=\frac{1}{(\tau+7)^2}\sin x$$

$$|g(\tau,x)-g(\tau,y)|\leq\frac{1}{49}|\sin x-\sin y|\leq\frac{1}{49}|x-y|$$

and

$$|g(\tau,x)|\leq\frac{1}{49}=\Psi(\tau)$$

Now

$$H\left[\frac{1}{\Gamma(l+1)}+\frac{\Gamma(k+3)}{\Gamma(l+1)[\Gamma(k+3)-6\beta\xi^{k+2}]}+\frac{\beta\Gamma(k+3)\xi^{k+l}}{\Gamma(k+l+1)[\Gamma(k+3)-6\beta\xi^{k+2}]}\right]$$

$$\approx 0.0049<1$$

Hence all the conditions of theorem 3.2 are satisfied, therefore the boundary value problem (4.3)-(4.4) has atleast one solution on [0,1].

V.CONCLUSION

In this paper, we presented a new type of nonlinear fractional three-point boundary value problem and find its existence and uniqueness of solutions by some well known fixed point theorems. Our results include related results existing in the literature.

REFERENCES

[1] I. Podlubny, Fractional differential equations (San Diego, Academic Press, 1999).

[2] R. Hilfer, Fractional calculus in physics (Singapore , World Scientific, 2000).

[3] A.A.Kilbas, H. M. Srivastva, J.J.Trujillo, Theory and applications of fractional differential equations (Elsevier, Amsterdam,2006).

[4] F. Mainardi, Fractional calculus and waves in linear viscoelasticity : An introduction to mathematicalmodel (Singapore, Imperial College Press, 2010).

[5] R. P. Agarwal, B. Andrade, C. Cuevas, Weighted pseudo-almost periodic solutions of a class of semilinear fractional differential equations,*Nonlinear Analysis: Real World Applications*, *11(5)*, 2010, 3532–3554.

[6] R. P. Agarwal, V. Lakshmikantham, J. J. Nieto, On the concept of solution for fractional differential equations with uncertainty, *Nonlinear Analysis: Theory, Methods & Applications*, *72(6)*, 2010, 2859–2862.

[7] B. Ahmad, Existence of solutions for irregular boundary value problems of nonlinear fractional differential equations, *Appl. Math. Lett., 23*, 2010, 390-394.

[8] B. Ahmad, S. Sivasundaram,On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order,*Appl. Math. Comput. 217*, 2010, 480-487.

[9] D. Delbosco, L. Rodino, Existence and uniqueness for a nonlinear fractional differential equation,*J. Math.Anal. Appl., 204,*1996, 609-625.

[10] J. Deng, L. Ma, Existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations,*Appl. Math. Lett., 23*, 2010, 676-680.

[11] X. Han, T. Wang, The existence of solutions for a nonlinear fractional multi-point boundary value problem at resonance,*Int. J. Differ. Equ. 2011*, 2011, Article ID 401803.

[12] W. Jiang, The existence of solutions to boundary value problems of fractional differential equations at resonance, *Nonlinear Anal., 74*, 2011, 1987-1994.