3rd International Conference on Research Developments in Applied Science, Engineering & Management
The Indian Council of Social Science Research (ICSSR)
Panjab University Campus, Chandigarh (India)
AEM–2018
19th August 2018 www.conferenceworld.in ISBN : 978-93-87793-43-9

Comprehensive Study of properties of fuzzy metric spaces

Happy Hooda[1], Archana Malik[2]

[1]Research Scholar, Dept. of Mathematics, MDU Rohtak
[2]Professor, Dept. of Mathematics, MDU Rohtak

Abstract

A metric space is a set of points and a prescribed quantitative measure of the degree of closeness of pairs of points in this space. The real number system and the coordinate plane of analytical geometry are familiar example of metric space. We also defined some topology on a fuzzy metric space and prove some known result of metric space including Baire's theorem for fuzzy metric spaces.

Introduction

Since the introduction of the concept of fuzzy sets by Zadeh in 1965, many authors have introduced the concept of fuzzy metric space in different ways. George and Veeramani. They showed also that every metric induces a fuzzy metric. The fuzzy version of Banach contraction principle was given by Grabiec [4] in 1988.

Now firstly we define fuzzy metric space -

Definition - A binary operation -

$$* : [0:1] \times [0:1] \rightarrow [0.1]$$

Is continuous t-norm if ([0:1]. $*$)

Is a topological monoid with unit 1 such that

$a * b \leq c * d$ whenever

$a \leq c$ and $b \leq d$. (a:

b : c: d $\in$ [0:1])

Definition -The 3-tuple (X:M:$*$) is said to be a fuzzy metric space if X is an arbitrary set $*$ is a continues t-norm and m is a fuzzy set on a

$X^2 \times [0:\infty)$ satisfying the following conditions:-

1. m (x:y:0) = 0
2. m (x: y : t) = 1 For all t > o if and only if X=Y
3. m (x: y : t) = m (y: x : t)
4. m (x: y : t) ∗ m (y: z : s) ≤ m (x : z : t +s)
5. m (x : y:.) : [o. ∞) → [0:1] is left continuous. x : y: z ∈ X and t : s >0

Definition- The 3-tuple (X:M ∗) is said to be a fuzzy metric space if X is an arbitrary set ∗ is a continuous t-norm and M is a fuzzy set on X^2 x (0:∞)

Satisfying the following conditions :-

1:- M (x : y : t) > 0
2:- M (x : y : t) = 1 if and only if x = y
3:- M (x : y : t) = M (y : x : t)
4:- M (x : y : t) ∗ M (y : z : s) ≤ M (x : z : t + s)
5:- M (x : y:.) : (0 : ∞) → [0:1] is continuous. x : y : z ∈ X and t : s > 0.

Definition- Let (F_x, μ_x) be a fuzzy metric space then an element A ∈ F_x is said to be an F- adherent point of a subset G of F_x if every F-open sphere with centre A contains at least one element of G.

F-adherent points are of two types:

(i) F- Limit Points or F- accumulations points:

(ii) F- isolated points

Definition - Let ($F_{X,}$ μ_x) be a fuzzy metric space. Then an element A∈F_x is said to be an F-limit point of F-accumulation point of a subset G of F_x if every F-open sphere with centre A contains at- least one element of G other then A.

An F-limit point of G is not necessarily a point of G.

Definition - An F-adherent point A of a subset G of F_x is called an F- isolated point if there exists at-least F-open sphere with centre A which contains no point element of G is an F- isolated point then G is called to be an F-isolated set.
Let ($F_{x,}\mu_x$) be a fuzzy metric space and let G ⊆ F_x Then the distance between A and G where A∈ F_x defined as

d(A,G)=inf. {μ_x (A,y):y∈G}.

Definition- As Hohle observed in [235] the fact that the topology associated to a probabilistic metric space is metrizable means that. from this topological point of view. Probabilistic metric spaces are always equivalent to ordinary metric spaces, and the problem of topologization of probabilistic metric space is not satlsfactorily solved. He proposed many-value3d topologies as suitable tools for this purpose. Hence in [251], we endowed George and veeramani's fuzzy metric (which has close relation to probabilistic metric) with many-valued structuesfuzzifying topology and fuzzifying uniformity. The aim of this paper is to go on studying the properties of George and Veeramani's fuzzy metric. We will give the concept of convewrgence degree and generalize the convergencwe and compactness theories in metric spaces to Veeramani's fuzzy metric space.

Properties of Fuzzy Metric Spaces

Since the value M(x,y,t) can be thought as the degree of the nearness between x and y with respect to t, in this section, we will give the definitions of degree convergence and study the relationship between them.

Let (X,M) be a fuzzy metric space, x ∈X and $\{x_n\}$ be sequence.

The degree to which $\{x_n\}$ converges to x is defined by

$$Con(\{x_n\},x) = \bigwedge_{\varepsilon>0} \bigvee_{N\in \mathcal{N}} \bigwedge_{n>N} M(x_n,x,\varepsilon).$$

The degree to which $\{x_n\}$ accumulates to x is defined by

$$Ad(\{x_n\},x) = \bigwedge_{\varepsilon>0} \bigvee_{N\in \mathcal{N}} \bigwedge_{n>N} M(x_n,x,\varepsilon).$$

The degree to which $\{x_n\}$ is a Cauchy sequence is defined by

$$Cauchy(\{x_n\},x) = \bigwedge_{\varepsilon>0} \bigvee_{N\in \mathcal{N}} \bigwedge_{n,m>N} M(x_n,x,\varepsilon).$$

Definition- In this section. We want to generalized the compactness in metric

ISBN : 978-93-87793-43-9

spaces to fuzzy setting according to the above convergence theory.

Let (X,M) be a fuzzy metric space. The degree to which (X,M) is compact is defined by

$$\text{Comp}(M) = \wedge_{\{x_n\}} \vee_{x \in X} Ad(\{x_n\},x)$$

The degree to which (X,M) is sequently compact is defined by

$$\text{Comp}(M) = \wedge_{\{x_n\}} \vee_{\{x_{nk}\}} \vee_{x \in X} Con(\{x_{nk}\},x).$$

Now we define some theorems related to fuzzy metric space.

Theorem- The element A of a fuzzy metric space $(F_x.\mu_x)$ is an F- adherent point of the subset G of F_x iff

$$d(A,G)=0$$

Proof: we have $d(A,G)= \inf. \{\mu_x(A.y): y \in G\}$ Therefore $d(A,G)=0 \Rightarrow$ every F-open sphere S (A,r) contains element of G. which implies A is an F-adherent point of G. conversely if A is an F-adherent point of G. then wither A is an F-isolated point of G case d(A.G) =O hence the theorem is proved.

Theorem Let X=R Define

ISBN : 978-93-87793-43-9

$$a * b = ab$$

and

$$m(x : y : t) = \exp.\left(\frac{|x-y|}{t}\right)^{-1}$$

for all x : y ∈ X and t ∈ (o: ∞). Then (x : m : ∗) is a fuzzy metric space.

Proof-

clearly m (x : y : t) = 1 if and only if x = y. If x = y.

1. m (x : y : t) = m (y : x : t)
2. To prove

   m (x : y : t). M (y: z : S) ≤ m (x : z : t + S) we know that

$$\because \; [x-z] \le \frac{t+S}{t}|x-y| + \frac{t+S}{S}|y-z|$$

i.e.

$$\frac{|x-z|}{t+S} \le \frac{|x-z|}{t} + \frac{|y-z|}{S}$$

therefore,

$$\exp\frac{|x-z|}{t+S} \le \exp.\frac{|x-y|}{S}\left(\exp\frac{|y-z|}{S}\right)$$

thus

m (x: y :t). m (y :z :s) ≤ m (x :z :t+S).

4. m (x :y: .) : (o:∞) → [0:1] is continuous. Hence (x: m: ∗) is a fuzzy metric space.

ISBN : 978-93-87793-43-9

Theorem: Every open ball is an open set.

Proof: Consider an open ball B (x : r : t) now

$Y \in B. (x : r : t) \Rightarrow m (x : y : t) > 1 - r$

Since $m (x : y : t) > 1\text{-}r$

We can find a to ; $0 < t_0 < t$ such that

$$m (x : y : t_0) > 1\text{-}r.$$

Let

$r_o = m (x : y : t_0) > 1\text{-}r$

since

$r_o > 1\text{-}r$. we can find a s;

$o < s < l$

such that

$r_o > 1 - s > 1 - r.$

now for a given r_o and S such that

$r_o > 1\text{-}S$ we can find r_1;

$o < r_1 < 1$

such that

$r_o * r_1 \geq 1\text{-}S.$

now consider the ball

$B (y. 1\text{-}r_1; t\text{-}t_0)$

We claim

$B (y : 1\text{-}r_1 : t - t_0) \subset B (x : r: t)$

Now

$z \in B (y : 1\text{-}r_1 : t\text{-}t_0) \Rightarrow m (y : z : t\text{-}t_0) > r_1$

ISBN : 978-93-87793-43-9

Therefore

$m (x : z : t) \geq m (x : y : t_1) * m (y : z : t - t_0)$

$\geq r_o * r_1 \geq 1\text{-}s$

$> 1\text{-}r$

Therefore $z \in B (x : r : t)$ and hence

$B (y : 1\text{-}r_1 : t\text{-}t_0) \subset B (x: r: t)$.

Theorem - Every fuzzy metric space is Hausdroff.

PROOF - Let (x : m : ∗) be the given fuzzy metric space.

Let x : y be two distinct points of x. then

$O < m (x : y : t) < l$

Let $m (x : y : t) = r$; for some r;

$O < r < l$.

For each r_o; $r < r_o < l$;

We can find a r_1 such that

$r_1 * r_1 \geq r_o$.

Now consider

The open balls $B (x : 1\text{-}r_1: \frac{1}{2} t)$

and

$B (y : 1\text{-}r_1: \frac{1}{2} t)$.

ISBN : 978-93-87793-43-9

Clearly

$$B\left(x : 1-r_1 : \tfrac{1}{2} t\right) \cap B\left(y : 1-r_1 : \tfrac{1}{2} t\right) = \emptyset$$

Theorem- Every compact subset A of a fuzzy metric space X is F-bounded.

Proof- Given A is a compact subset of X. Fix

$t > o$ and $o < r < l$.

consider an open cover

$\{ B(x : r : t) : x \in A \}$ of A.

Since A is compact; there exist

$x_1 : x_2 : \ldots\ldots\ldots\ldots x_n \in A$.

Such that

$A \subseteq \cup B(x : r : t)$.

Let $x : y : \in A$.

Then

$x \in B(x_i : r : t)$

and $y \in B(x_i : r : t)$

for some $i : j$.

therefore $m(x, x_i, t) > 1-r$

and $m(y : x_j : t) > 1-r$

now let

$\alpha = \min \{ m(x_i : x_j : t) : \leq i : j \leq n \}$

Then $\alpha > o$

Now

$m(x : y : 3t) \geq m(x : x_i : t) * m(x_i : x_j : t)$

$* \, m (x_j : y : t)$

$\geq (1\text{-}r) * (1\text{-}r) * \alpha$.

Taking $t' = 3t$ and

$(1\text{-}r) * (1\text{-}r) * \alpha > 1\text{-}S : o < s < 1$

We have

$m (x : y : t') > 1\text{-}S$

for all $x : y \in A$

hence A is F- bounded.

Baire's theorem- let X be a complete fuzzy metric space. Then the intersection of a countable number of dense open sets in dense.

Proof- Let X be the given complete fuzzy metric space. Let B_o be a non-empty open set.

Let

$D_1 : D_2 : D_3$ .... be dense open sets in X since D_1 is dense in X:

$B_o \cap D_1 = \emptyset$.

Let

$X_1 \in B_o \cap D_{1.}$

Since $B_o \cap D_1$. is open. There exist

$O < r_1 <1$:

t_1 >o such that

$B(x_1 : r_1 : t_1) \subset B_o \cap D_1$.

Choose $r_{1'} < r_1$

and $t_{1'} = \min \{t_1 : 1\}$

such that

$B [x_1 : r_1' : t_1'] \subset B_o \cap D_1$.

Let

$B_1 = B (x_1 : r_1' : t_i')$.

Since D_2 in dense in X:

$B_1 \cap D_2 \neq \emptyset$

Let

$X_2 \in B_1 \cap D_2$

Since $B_1 \cap D_2$ is open there exits $o < r_2 < \frac{1}{2}$ and $t_2 > o$

Such that

$B (x_2 : r_2 : t_2) \subset B \cap D_2$.

Choose

$r_2' < r_2$

ISBN : 978-93-87793-43-9

and $t_2' = \min \{ t_2 : \frac{1}{2} \}$

Such that $B [x_2 : r_{2'} , t_{2'}] \subset B_1 \cap D_2$.

Let

$$B_2 = B (x_2 : (x_2 : r_2' : t_2')$$

Similarly proceeding by induction we can find a $x_n \in B_n$-1∩ Dn.

For a given $t > o : \varepsilon > o$:

Choose n_o such that

$$\frac{1}{n_o} < t \text{ and } \frac{1}{n_o} < \in$$

then for $n \geq n_o$; $m \geq n$.

$$m (x_n : x_m : t) \geq M (x_n : x_m : 1/n)$$

$$\geq 1- \frac{1}{n}$$

$$\geq 1- \frac{1}{n}$$

$$\geq 1 - \in$$

Therefore $\{ x_n \}$ is a Cauchy sequence since X is complete.

$$X_n \rightarrow x.$$

But $x_k \in B [x_n : r_n' : t_n']$

For all $K \geq n$. and by the previous result $B [x_n : r_n' : t_n']$ is a closed set.

Hence

$$x \in B [x_n : r_n' : t_n'] \subset B_{n-1} \cap Dn$$

for all n.

therefore

$$B_o \cap \left(\bigcap_{n=1}^{\infty} Dn\right) \neq \emptyset.$$

Hence $\bigcap_{n=1}^{\infty} Dn$ dense in X.

Conclusion

In this paper, the properties of fuzzy metric spaces have been studied and verified with some theorems and proofs. The authors have studied fuzzy metric spaces in connection with several kinds of boundedness properties related to selection principles and studied already in other mathematical structures, such as uniform spaces and topological groups.

References

[1]. George, P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets Syst. 64 (1994) 395-399.
[2]. V. Gregori, S. Romaguera, Some properties of fuzzy metric spaces, Fuzzy Sets Syst. 115 (2000) 485-489.
[3]. O. Kaleva, S. Seikkala, On fuzzy metric spaces, Fuzzy Sets Syst. 12 (1984) 215–229.
[4]. Lj.D.R. Kocinac, Selection principles in uniform spaces, Note Mat. 22 (2003 ˇ /2004) 127–139.
[5]. J. Kramosil, J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetika 11 (1975) 326–334.
[6]. J.H. Park, Intuitionistic fuzzy metric spaces, Chaos, Solitons & Fractals 22 (2004) 1039–1046.
[7]. R. Saadati, Notes to the paper ”Fixed points in intuitionistic fuzzy metric spaces” and its generalization to L-fuzzy metric spaces, Chaos, Solitons & Fractals 35 (2008) 176–180.
[8]. S. Kumar, Common fixed point theorem in fuzzy 2-metric spaces, Uni. Din. Bacau. Studii Si Cercetiri Sciintifice, Serial: Mathematical Nr, 18(2008), 111-116.
[9]. L.A. Zadeh, Fuzzy sets, Information and Control, 8(1965), 338-353.