Fixed Point Theorem in Probabilistic Metric Spaces

Piyush Kumar Tripathi, Alok Agrawal
[1,2] *Amity School of Applied Sciences, Amity University, Lucknow Campus, Uttar Pradesh, India*

Abstract
Since every metric space is a probabilistic metric space with natural distribution function without holding converse and therefore results given by us in section 2 are true in more generalized setting of probabilistic metric space. In this paper the concept of dual contraction mappings has been defined in probabilistic metric space and using this we have obtained a number of fixed point theorems in the light of some new contractive conditions.
Keywords: metric space, Probabilistic metric space

1. INTRODUCTION:

The Banach fixed point theorem guarantees the existence of unique fixed point under a contraction mapping on a complete metric space. A similar theorem does not hold in a complete Menger Probabilistic metric space. The problem is that the triangular function in such spaces is not enough to guarantee that the sequence of iterates of a point under a certain map is Cauchy sequence. Two different approaches have been pursued. One is to identify those triangle functions which guarantee that the sequence of iterates is a Cauchy sequence. The other is to modify the original definition of contraction map. In probabilistic metric space fixed point theorems using the concept of contraction mapping have been studied by many authors, namely Jeong G. S.[1], B.D.Pant [2], and S. L. Singh [5] etc. Infact Sehgal [4] introduced the above concept for first time in a probabilistic metric space and proved several significant results.Jeong G. S.[1], B.D.Pant [2], and S. L. Singh [5] etc contributed some more results which can be seen in [1], [2], and [5]. These results and paper of S.L. Singh, B.D.Pant [5] and B.E.Rhoads [3] prompted us to go for the definition of dual contraction in probabilistic metric space, which is independent of contraction, defined by Sehgal and others. We have given more results using dual contraction in section 2 of this paper. Before presenting our results we mention below the definition and important results of above authors. By motivation of paper of Piyush Kumar Tripathi we have also proved some more results of fixed point theorem using different contractive conditions in section 3 of this paper.

PRELIMINARIES:

In this section we recall some basic definition and results of probabilistic metric space. For more details we refer the reader to [1], [3], [5] and [6].

2.1.1 DEFINITION: A mapping $f : R \to R^+$ is called a distribution function if it is non decreasing, left continuous and inf$f(x) = 0$, sup $f(x) = 1$.

We shall denote by L the set of all distribution function. The specific distribution function $H \in L$ is defined by

$$H(x) = 0 \ , \ x \le 0$$
$$= 1 \ , \ x > 0$$

2.1.2 DEFINITION: A probabilistic metric space (PM space) is an ordered pair (*X*,*F*), *X* is a nonempty set and $F : X \times X \to L$ is mapping such that,

(I) $F_{p,q}(x) = 1 \ \forall \ x > 0$ iff $p = q$

(II) $F_{p,q}(0) = 0$

(III) $F_{p,q} = F_{q,p}$

(IV) $F_{p,q}(x) = 1$, $F_{q,r}(y) = 1 \Rightarrow F_{p,r}(x+y) = 1$

We note that $F_{p,q}(x)$ is value of the function $F_{p,q} = F(p,q) \in L$ at $x \in$ □

2.1.3 DEFINITION: A mapping $t : [01] \times [01] \to [01]$ is called t-norm if it is non decreasing, commutative, associative and $t(a,1) = a \quad \forall \ a \in [01]$.

2.1.4 DEFINITION: A Menger PM space is a triple (*X*,*F*;*t*) where (*X*,*F*) is a PM space and t is t-norm such that $F_{pr}(x+y) \ge t\left(F_{pq}(x), F_{qr}(y)\right) \ \forall \ x, y \ge 0$. In [8] it is seen that if (*X*,*F*;*t*) is Menger Probabilistic metric space with $\sup t\left(x,x\right) = 1$, $0 < x < 1$. Then (*X*, *F*; *t*) is a Hausdorff topological space in the topology T induced by the family of $(\in, \lambda)$ neighborhoods $\{U_p(\in,\lambda) : p \in X, \in > 0, \lambda > 0\}$ where

$$U_p(\in,\lambda) = \{x \in X : F_{x,p}(\in) > 1 - \lambda\}.$$

2.1.5 DEFINITION: A sequence $\{p_n\}$ in *X* is said to converges $p \in X$ iff $\forall \ \varepsilon > 0$ and $\lambda > 0$, $\exists$ an integer M such that $F_{p_n p}(\varepsilon) > 1 - \lambda \ \forall n \ge M$. Again $\{p_n\}$ is a Cauchy sequence if $\forall \in > 0$ and $\lambda > 0 \ \exists$ an integer *M* such that $F_{p_n p_m}(\varepsilon) > 1 - \lambda \ \forall \ m, n \ge M$.

2.1.1 Lemma [5]: Suppose $\{p_n\}$ is a sequence in Menger space (X, F; *t*), where *t* is continuous and $t(x,x) \ge x \quad \forall \quad x \in [01]$. If $\exists \ k \in (0,1)$ s.t. $\forall \ x > 0$ and positive integer *n* such that $F_{p_n p_{n+1}}(kx) \ge F_{p_{n-1} p_n}(x)$, then $\{p_n\}$ is a Cauchy sequence.

REMARK: The above lemma 1.1 can also be written as " Suppose $\{p_n\}$ is a sequence in

Menger space (X, F; t), where t is continuous and $t(x,x) \geq x \quad \forall \quad x \in [01]$. If $\exists\, k > 1$ such that $\forall\, x > 0$ and positive integer n, $F_{p_{n-1},p_n}(kx) \leq F_{p_n,p_{n+1}}(x)$, then $\{p_n\}$ is a Cauchy sequence". This is possible because if $k > 1$ and $F_{p_{n-1},p_n}(kx) \leq F_{p_n,p_{n+1}}(x)$, then

$F_{p_{n-1},p_n}(x) \leq F_{p_n,p_{n+1}}(\frac{1}{k}x) = F_{p_n,p_{n+1}}(k'x) \Rightarrow F_{p_n,p_{n+1}}(k'x) \geq F_{p_{n-1},p_n}(x)$ where $k' = \frac{1}{k} \in (0,1)$ so by lemma 2.1.1 $\{p_n\}$ is a Cauchy sequence.

2.1.6 DEFINITION [5]: Let (X,F) be a PM space and $f : X \to X$ be a mapping defined on X.Then f is said to contraction if $\exists\ k \in (0\ 1)$ s.t. $\forall\ p,q \in X$, and for all $x > 0$, $F_{f(p)f(q)}(kx) \geq F_{pq}(x)$.

2.1.1 THEOREM [5]: Every contraction mapping has at most one fixed point .

2.1.2 Lemma [5]: If (X,d) is a metric space, then the metric d induces a mapping $F : X \times X \to X$ defined by $F_{p,q}(x) = H\left(x - d(p,q)\right)$, $p,q \in X$ and $x \in$ □ .

Through this mapping every metric space can be considered as probabilistic metric space

2.2.1 THEOREM [6]: Let (X,F,t) be a complete Menger space and $t(x,x) \geq x \ \ \forall\, x \in [0,1]$. If $f : X \to X$ continuous function and $\{p_n\}$ is a Cauchy sequence defined by $p_n = fp_{n-1}$ converges to $p \in X$. Then p is a fixed point of f.

2.2.2 THEOREM [6]: Let (X,F,t) be a complete Menger space with two mapping $f, g : X \to X$ such that,

(I) $F_{fpfq}(x) \geq F_{gpgq}(x) \quad \forall\ p,q \in X, x > 0$

(II) f is continuous.

(III) g is contraction.

Then f has a unique fixed point.

2.2.1 DEFINITION: Let (X,F,t) be a Menger space. A mapping $f : X \to X$ is called dual contraction if $\exists\ k > 1$ such that $F_{fpfq}(kx) \leq F_{pq}(x)$, $x > 0$

3. Main Result

3.1 THEOREM: Suppose (X,F,t)be complete Menger space. Suppose $f : X \to X$ is surjective
mapping satisfying the condition of dual contraction. If $\exists\ k > 1$ such that $F_{fpfq}(kx) \leq F_{pq}(x)$,

x>0.Then$f$ has a unique fixed point.

PROOF: If $p \neq q$ and $fp = fq$ then $1 \leq F_{pq}(x)$ which is not possible because $F_{pq}(x) < 1$,

so$f$ is one to one that is f is one- one onto mapping. Let $f^{-1} = g$. Then by

dual contraction $F_{pq}(kx) \leq F_{gpgq}(x)\ \forall\ p,q \in X, x > 0$

$$F_{gpgq}\left(\frac{1}{k}x\right) = F_{gpgq}(k'x) \geq F_{pq}(x),\ \frac{1}{k} = k',\ 0 < k' < 1$$

Then g is contraction and satisfying all the condition of theorem 2.2.2 so

$\exists\ p_0 \in X$ such that

$$g(p_0) = p_0$$

$$f^{-1}(p_0) = p_0 \Rightarrow f(p_0) = p_0$$

so p_0 is a unique fixed point of f.

3.1 Lemma: Let (X,F,t)be a Menger space, where t is continuous. If $\exists\ k > 1$such that $F_{fpf^2p}(kx) \leq F_{pfp}(x)$, $x > 0$. Suppose $f : X \to X$ is surjective mapping then $\exists$a Cauchy sequence in X.

PROOF: Since f is onto s o for $p_0 \in X$, $\exists\ p_1 \in X$ such that $f(p_1) = p_0$. Construct a sequence $\{p_n\}$ as $f(p_n) = p_{n-1}$, $n = 1,2$........... then

$$F_{fp_n f^2 p_n}(kx) \leq F_{p_n fp_n}(x)$$

$$F_{p_{n-1}p_{n-2}}(kx) \leq F_{p_n p_{n-1}}(x)$$

$$F_{p_n p_{n-1}}\left(\frac{1}{k}x\right) \geq F_{p_{n-1}p_{n-2}}(x)$$

$$F_{p_n p_{n-1}}(k'x) \geq F_{p_{n-1}p_{n-2}}(x)$$

Since $\frac{1}{k} = k' \in (0,1)$ so by lemma 2.1.1 $\{p_n\}$ is a Cauchy sequence

3.2 THEOREM: Let (X,F,t) be a complete Menger space and $f : X \to X$ is continuous and onto mapping satisfying the property of lemma 2.2.1. Then f has a fixed point.

PROOF: By lemma 2.2.1 $\exists$a Cauchy sequence $\{p_n\}$ in X. since X is complete

so $p_n \to p \in X$ then by theorem 2.2.1 p is a fixed point of f.

3.3THEOREM: Let (X,F,t) be complete Menger space and $f : X \to X$ is continuous onto mapping. If $\exists k$> 1such that $F_{fpfq}(kx) \leq \min\{F_{pfp}(x), F_{qfq}(x), F_{pq}(x)\}$. Then f has a fixed point.

PROOF: Let $p_0 \in X$, as lemma 2.2.1 we can construct a sequence as

$p_{n-1} = f(p_n)$, $n = 1, 2....$

$$F_{p_{n-1}p_n}(kx) = F_{fp_n fp_{n+1}}(kx) \le \min\{F_{p_n p_{n-1}}(x), F_{p_{n+1}p_n}(x), F_{p_n p_{n+1}}(x)\}$$

$$F_{p_{n-1}p_n}(kx) \le \min\{F_{p_n p_{n-1}}(x), F_{p_n p_{n+1}}(x)\}$$

Since F is non-decreasing function and k> 1so kx>x**.** Then

$F_{p_{n-1}p_n}(kx) \le F_{P_n p_{n+1}}(x)$ i.e

$$F_{p_n p_{n+1}}\left(\frac{1}{k}x\right) = F_{p_n p_{n+1}}(k'x) \ge F_{p_{n-1}p_n}(x) \text{ , } \frac{1}{k} = k' \in (0,1)$$

So $\{p_n\}$ is a Cauchy sequence. Since X is completeso $p_n \to p \in X$, then by theorem 2.2.1 p is a fixed point of f.

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