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SOME ADVANCED CONTINUOUS MAPS AND THEIR CHARACTERIZATIONS IN BINARY TOPOLOGICAL STRUCTURES

[1]Pooja Amol Dhole, [2]Dr. Nazir Ahmad Ahengar

[1] Dr. D. Y. Patil Technical Campus Varale Talegaon. Pune-410507, Maharashtra (India)

[2] School of Engineering and Technology, Pimpri Chinchwad University, Pune-412106, Maharashtra (India)

[1]pooja.varade23@pcu.edu.in

[2]nazir.ahengar@pcu.edu.in

Abstract:

A binary topology from U to V is a structure defined on ordered pairs of subsets that satisfies axioms analogous to those of classical topology. This notion, introduced by S. Nithyanantha Jothi and P. Thangavelu, provides a unified framework for studying relationships between two sets. In this paper, we introduce the concept of some advance continuities in binary topological spaces. Various forms of continuity, including semi, pre, α and β-continuity, are examined. Fundamental properties and interrelationships among these mappings are established with the help of suitable examples.

Keywords: Binary continuity, advanced binary topology, advanced binary continuity

1. INTRODUCTION

Topology is a fundamental branch of mathematics concerned with the study of continuity, convergence, and abstract structures. Classical topology deals with collections of subsets of a single set satisfying certain axioms, forming the basis for analyzing continuous mappings and related properties. Standard references such as Engelking [4] and Kelley [9] provide a comprehensive treatment of these ideas. Over time, researchers have introduced several advanced concepts to extend the scope of topology. Notably, Levine [2], [3] introduced semi-open sets and advanced closed sets, which have proven useful in studying weaker forms of

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continuity. These developments have significantly enriched topological theory and motivated further investigations into advanced structures [12], [13].

In recent years, attention has shifted toward extending topological concepts to frameworks involving more than one underlying set. In this direction, Nithyanantha Jothi and Thangavelu [5]–[7] introduced the concept of binary topology. A binary topology from a set AA to a set BB is a binary structure consisting of ordered pairs of subsets that satisfy axioms analogous to those of classical topology. This approach provides a unified setting in which subsets of two distinct sets can be studied simultaneously. As a result, binary topological spaces and corresponding notions such as binary continuity and separation axioms have been developed and investigated. These structures allow for a richer interaction between sets and open new avenues for extending classical results.

To further generalize these ideas, the notion of advanced binary topology has been introduced, where certain axioms of binary topology are relaxed to obtain a more flexible framework [1]. In such spaces, advanced binary open sets are defined without requiring closure under finite intersections, thereby extending the applicability of the theory. This generalization makes it possible to study broader classes of mappings and to analyze their properties in a more general setting. In particular, advanced binary topological spaces provide a natural environment for extending various types of continuity beyond the classical and binary cases.

The concept of continuity plays a central role in topology, and its generalizations have been widely studied. Different forms of continuity, such as semi-continuity, pre-continuity, α-continuity, and β-continuity, have been introduced to capture finer variations in the behavior of functions. These notions are closely related to corresponding types of advanced open sets and have been investigated extensively in classical and advanced topological settings [12], [13]. Extending these ideas to advanced binary topological spaces leads to the study of advanced binary continuous mappings and their various forms.

The purpose of this paper is to introduce and study the concept of some advanced continuous maps in binary topological spaces. We define and analyze different types of these continuous mappings, including advanced binary semi-continuous, pre-continuous, α-continuous, and β-continuous functions. The relationships among these classes of mappings are examined in

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detail, and several examples are provided to illustrate their independence and hierarchical structure. These results contribute to a deeper understanding of advanced binary topological spaces and their associated mappings.

The organization of the paper is as follows. Section 2 is devoted to the basic concepts and preliminary results related to advanced binary topology. In Section 3, various types of advanced binary continuous mappings are introduced and studied, and their interrelationships are established. Throughout the paper, $P(A)$ denotes the power set of a set A.

2. PRELIMINARIES

Definition 2.1: Let X and Y be any two non-empty sets. The authors defined that a binary topology from X to Y is a binary structure $M \subseteq \rho(X) \times \rho(Y)$ that satisfies the following axioms.

1) $(\emptyset, \emptyset)$ and $(X, Y) \in M$,
2) $(A_1 \cap A_2, B_1 \cap B_2) \in M$ whenever $(A_1, B_1) \in M$ and $(A_2, B_2) \in M$,
3) If $\{(A_\alpha, B_\alpha): \alpha \in \Delta\}$ is a family of members of M, then $((\cup_{\alpha\in\Delta} A_\alpha, \cup_{\alpha\in\Delta} B_\alpha)) \in M$.

Definition 2.2: If M is a binary topology from X to Y then the triplet (X, Y, M) is called a binary topological space and the members of M are called the binary open subsets of the binary topological space (X, Y, M). The elements of $X \times Y$ are called the binary points of the binary topological space (X, Y, M).
If $Y = X$ then M is called a binary topology on X in which case we write (X, M) as a binary space.

Definition 2.3: Let (X, Y, M) be a binary topological space and $A \subseteq X, B \subseteq Y$. then (A, B) is called binary closed in (X, Y, M) if $(X\backslash A, Y\backslash B) \in M$.

Definition 2.4: Let A to B be any two non-empty sets. An advanced binary topology from A to B be a binary structure $\mathcal{T}_A \subseteq \mathcal{P}(A) \times \mathcal{P}(B)$ that satisfies the following axioms:

i) $(\emptyset, \emptyset)$ and $(A, B) \in \mathcal{T}_A$.

ii) If $\{(A_\alpha, B_\alpha)\ \alpha \in \Delta\}$ is a family of members of $\mathcal{T}_A$ then $(\cup_{\alpha\in\Delta} A_\alpha, \cup_{\alpha\in\Delta} B_\alpha) \in \mathcal{T}_A$.

If $\mathcal{T}_A$ is an advanced binary topology from A to B then the triplet $(A, B, \mathcal{T}_A)$ is called an

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advanced binary topological space and the members of $\mathcal{T}_{\text{Ā}}$ are called the advanced binary open subsets of the advanced binary topology $(A, B, \mathcal{T}_{\text{Ā}})$ The elements of $A \times B$ are called the advanced binary points of advanced binary topology $(A, B, \mathcal{T}_{\text{Ā}})$

Definition 2.5: Let $(A, B, \mathcal{T}_{\text{Ā}})$ be an advanced binary topological space and $A_1 \subseteq A$, $B_1 \subseteq B$. Then (A_1, B_1) is advanced binary closed in $(A, B, \mathcal{T}_{\text{Ā}})$ if $(A \backslash A_1, B \backslash B_1) \in \mathcal{T}_{\text{Ā}}$.

Proposition 2.1: Let $(A, B, \mathcal{T}_{\text{Ā}})$ be an advanced binary topological space. Then

i) $(\emptyset, \emptyset)$ and $(A, B))$ are advanced binary closed sets.

ii) If $\{(A_\alpha, B_\alpha): \alpha\}$ is a family of advanced binary closed sets, then $(\bigcap_\alpha A_\alpha, \bigcap_\alpha B_\alpha)$ is advanced binary closed.

Definition 2.6: Let $(A, B, \mathcal{T}_{\text{Ā}})$ be an advanced binary topological space and $(A_1, B_1) \subseteq (A, B)$. Let $(A_1, B_1)^{1^*} = \bigcap\{A_\alpha: (A_\alpha, B_\alpha)$ is advanced binary closed and $(A_1, B_1) \subseteq (A_\alpha, B_\alpha)\}$ and Let $(A_1, B_1)^{2^*} = \bigcap\{B_\alpha: (A_\alpha, B_\alpha)$ is advanced binary closed and $(A_1, B_1) \subseteq (A_\alpha, B_\alpha)\}$. Then $\left((A_1, B_1)^{1^*}, (A_1, B_1)^{2^*}\right)$ is advanced binary closed and $(A_1, B_1) \subseteq \left((A_1, B_1)^{1^*}, (A_1, B_1)^{2^*}\right)$ The ordered pair $\left((A_1, B_1)^{1^*}, (A_1, B_1)^{2^*}\right)$ is called advanced binary closure of (A_1, B_1) and is denoted $\text{Ā}cl(A_1, B_1)$ in the advanced binary topology $(A, B, \mathcal{T}_{\text{Ā}})$. Where $(A_1, B_1) \subseteq (A, B)$.

Remark 2.1: In an advanced binary topological space $(A, B, \mathcal{T}_{\text{Ā}})$. If $(A_1, B_1) \subseteq (A, B)$, then $\text{Ā}cl(A_1, B_1)$ is smallest advanced binary closed set Containing (A_1, B_1).

Proposition 2.2: Let $(A_1, B_1) \subseteq (A, B)$. Then (A_1, B_1) is advanced binary closed in $(A, B, \mathcal{T}_{\text{Ā}})$ iff $(A, B) = \text{Ā}cl(A_1, B_1)$.

Definition 2.7: Let $(A, B, \mathcal{T}_{\text{Ā}})$ be an advanced binary topological space and $(A_1, B_1) \subseteq (X, Y)$. Let $(A_1, B_1)^{1^0} = \bigcup\{A_\alpha: (A_\alpha, B_\alpha)$ is advanced binary open and $(A_\alpha, B_\alpha) \subseteq (A_1, B_1)\}$ and Let $(A_1, B_1)^{2^0} = \bigcup\{B_\alpha: (A_\alpha, B_\alpha)$ is advanced binary open and $(A_\alpha, B_\alpha) \subseteq (A_1, B_1)\}$.Then $\left((A_1, B_1)^{1^0}, (A_1, B_1)^{2^0}\right)$ is advanced binary open and $\left((A_1, B_1)^{1^0}, (A_1, B_1)^{2^0}\right) \subseteq (A_1, B_1)$. The ordered pair $\left((A_1, B_1)^{1^0}, (A_1, B_1)^{2^0}\right)$ is called advanced binary interior of (A_1, B_1) and is denoted by $\text{Ā}int\,(A_1, B_1)$.

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Proposition 2.3: In an advanced binary topological space $(A, B, \mathcal{T}_{\bar{A}})$ if $(A_1, B_1) \subseteq (A, B)$, then $\bar{A}\text{int}\,(A_1, B_1)$is largest advanced binary open set Contained in (A_1, B_1).

Proposition 2.4: Let $(A_1, B_1) \subseteq (A, B)$. Then (A, B) is advanced binary open in $(\mathbf{A}, \mathbf{B}, \mathcal{T}_{\bar{A}})$ iff $(A_1, B_1) = \bar{A}\text{int}\,(A_1, B_1)$.

Definition 2.8: Let $(\mathbf{A}, \mathbf{B}, \mathcal{T}_{\bar{A}})$ be an advanced binary topological space and let $\lambda \subseteq A \times B$ be a binary set. Then:

i) λ is said to be a binary semi-open set if $\lambda \subseteq \text{cl}_{\mathcal{T}_{\bar{A}}}(\text{int}_{\mathcal{T}_{\bar{A}}}(\lambda))$.

ii) λ is said to be a binary pre-open set if $\lambda \subseteq \text{int}_{\mathcal{T}_{\bar{A}}}(\text{c}_{\mathcal{T}_{\bar{A}}}(\lambda))$

iii) λ is said to be a binary α-open set if $\lambda \subseteq \text{int}_{\mathcal{T}_{\bar{A}}}(\text{cl}_{\mathcal{T}_{\bar{A}}}(\text{int}_{\mathcal{T}_{\bar{A}}}(\lambda)))$.

iv) λ is said to be a binary β-open set if $\lambda \subseteq \text{cl}_{\mathcal{T}_{\bar{A}}}(\text{int}_{\mathcal{T}_{\bar{A}}}(\text{cl}_{\mathcal{T}_{\bar{A}}}(\lambda)))$.

Proposition 2.5: Let $(\mathbf{A}, \mathbf{B}, \mathcal{T}_{\bar{A}})$ be an advanced binary topological space. Then

i) Every advanced binary open set is an advanced binary semi-open set,

ii) Every advanced binary open set is an advanced binary pre-open set,

iii) Every advanced binary open set is an advanced binary α-open set,

iv) Every advanced binary open set is an advanced binary β-open set.

Proposition 2.6: Let $(\mathbf{A}, \mathbf{B}, \mathcal{T}_{\bar{A}})$ be an advanced binary topological space. Then the following statements hold:

i) Every advanced binary α-open set is an advanced binary semi-open set.

ii) Every advanced binary α-open set is an advanced binary pre-open set.

iii) Every advanced binary semi-open set is an advanced binary β-open set.

iv) Every advanced binary pre-open set is an advanced binary β-open set.

v) Every advanced binary α-open set is an advanced binary β-open set.

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3. ADVANCED FORMS OF BINARY CONTINUOUS MAPS.

Definition 3.1: Let $(A, B, \mathcal{T}_{A})$ be an advanced binary topological space and (Z, T) be an advanced topological space. A function $F: Z \rightarrow A \times B$ is called advanced binary continuous at $z \in Z$ if for any advanced binary open set $(A_1, B_1) \in (A, B, \mathcal{T}_{A})$ with $F(z) \in (A_1, B_1)$ then there exists an advanced open set Z_1 in (Z, T) such that $z \in Z_1$ and $F(Z_1) \subseteq (A_1, B_1)$ The function F is called advanced binary continuous if it is advanced binary continuous at each $z \in Z$.

Remark 3.1: Let $(A, B, \mathcal{T}_{A})$ be an advanced binary topological space and (Z, T) be an advanced topological space. Let $F: Z \rightarrow A \times B$ be a map. Then F is called advanced binary continuous map if $F^{-1}(A_1, B_1)$ is advanced open in (Z, T) for every advanced binary open set (A_1, B_1) in $(A, B, \mathcal{T}_{A})$.

Definition 3.2: Let $(A, B, \mathcal{T}_{A})$ be an advanced binary topological space and (Z, T) be an advanced topological space. A function $F: Z \rightarrow A \times B$ is called advanced binary semi-continuous if $F^{-1}(A_1, B_1)$ is advanced semi open in (Z, T) for every advanced binary open set (A_1, B_1) in $(A, B, \mathcal{T}_{A})$.

Example 3.1: Let $Z = \{a, b, c, d\}$, $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Then $T = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, Z\}$ and $\mathcal{T}_{A} = \{(\emptyset, \emptyset), (\{a_1\}, \{b_1\}), (\{a_2\}, \{b_1\}), (\{a_1, a_2\}, \{b_1\}), (A, B)\}$. Clearly T is an advanced topology on Z and $\mathcal{T}_{A}$ is advanced binary topology from A to B. Define $F: Z \rightarrow A \times B$ by $F(a) = (a_1, b_1), F(b) = F(c) = F(d) = (a_2, b_1)$,Now $F^{-1}(\emptyset, \emptyset) = \emptyset$, $F^{-1}(\{a_1\}, \{b_1\}) = \{a\}$, $F^{-1}(\{a_2\}, \{b_1\}) = \{b, c, d\}$, and $F^{-1}(\{a_1, a_2\}, \{b_1\}) = \{a, b, c, d\} = Z$, This shows that the inverse image of every advanced binary open sets in $(A, B, \mathcal{T}_{A})$ is advanced semi-open in (Z, T). Hence F is advanced binary semi-continuous.

Definition 3.3: Let $(A, B, \mathcal{T}_{A})$ be an advanced binary topological space and (Z, T) be an advanced topological space. A function $F: Z \rightarrow A \times B$ is called advanced binary pre-continuous if $F^{-1}(A_1, B_1)$ is advanced pre-open in (Z, T) for every advanced binary open set (A_1, B_1) in $(A, B, \mathcal{T}_{A})$.

Example 3.2: From Example 3.1, the function F is advanced binary pre-continuous.

Definition 3.4: Let $(A, B, \mathcal{T}_{A})$ be an advanced binary topological space and (Z, T) be an advanced topological space. A function $F: Z \rightarrow A \times B$ is called advanced binary α-

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continuous if $F^{-1}(A_1, B_1)$ is advanced α open in (Z, T) for every advanced binary open set (A_1, B_1) in $(A, B, \mathcal{T}_A)$.

Example 3.3: From Example 3.1, the function F is advanced binary α -continuous.

Definition 3.5: Let $(A, B, \mathcal{T}_A)$ be an advanced binary topological space and (Z, T) be an advanced topological space. A function $F: Z \to A \times B$ is called advanced binary β -continuous if $F^{-1}(A_1, B_1)$ is advanced β open in (Z, T) for every advanced binary open set (A_1, B_1) in $(A, B, \mathcal{T}_A)$.

Example 3.4: From Example 3.1, the function F is advanced binary β -continuous.

Proposition 3.1: Every advanced binary continuous function on an advanced binary topological space is advanced binary semi-continuous.

Proof: Let (A_1, B_1) be an advanced binary open set in $(A, B, \mathcal{T}_A)$. Since $F: (Z, \mathcal{T}) \to (A, B, \mathcal{T}_A)$ is an advanced binary continuous function, it follows that $F^{-1}(A_1, B_1)$ is an advanced open set in $(Z, \mathcal{T})$. It is known that every advanced open set is an advanced semi-open set. Therefore, $F^{-1}(A_1, B_1)$ is an advanced semi-open set in $(Z, \mathcal{T})$. Hence, the inverse image of every advanced binary open set in $(A, B, \mathcal{T}_A)$ is an advanced semi-open set in $(Z, \mathcal{T})$. Hence, F is an advanced binary semi-continuous.

Remark 3.2: The converse of the Proposition 3.1 need not be true as shown in Example 3.5.

Example 3.5: Let $Z = \{a, b, c, d\}$, $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Then $T = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, Z\}$ and $\mathcal{T}_A = \{(\emptyset, \emptyset), (\{a_1\}, \{b_1\}), (\{a_2\}, \{b_1\}), (\{a_1, a_2\}, \{b_1\}), (A, B)\}$. Clearly T is an advanced topology on Z and $\mathcal{T}_A$ is advanced binary topology from A to B. Define $F: Z \to A \times B$ by $F(a) = (a_1, b_1), F(b) = F(c) = F(d) = (a_2, b_1)$, Now $F^{-1}(\emptyset, \emptyset) = \emptyset$, $F^{-1}(\{a_1\}, \{b_1\}) = \{a\}$, $F^{-1}(\{a_2\}, \{b_1\}) = \{b, c, d\}$, and $F^{-1}(\{a_1, a_2\}, \{b_1\}) = Z$, This shows that the inverse image of every advanced binary open set in $(A, B, \mathcal{T}_A)$ is advanced semi-open in (Z, T). Hence F is advanced binary semi-continuous but not advanced binary continuous because $\{b, c, d\}$ is advanced semi-open but not advanced open in (Z, T).

Proposition 3.2 : Every advanced binary continuous function on an advanced binary topological space is advanced binary pre-continuous.

Proof: Let (A_1, B_1) be an advanced binary open set in $(A, B, \mathcal{T}_A)$. Since $F: (Z, \mathcal{T}) \to (A, B, \mathcal{T}_A)$ is an advanced binary continuous function, it follows that $F^{-1}(A_1, B_1)$ is an advanced open set in $(Z, \mathcal{T})$. It is known that every advanced open set is an advanced pre-open set.

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Therefore, $F^{-1}(A_1, B_1)$ is an advanced pre-open set in $(Z, \mathcal{T})$. Hence, the inverse image of every advanced binary open set in $(A, B, \mathcal{T}_{A})$ is an advanced pre-open set in $(Z, \mathcal{T})$. Thus, F is an advanced binary pre-continuous.

Remark 3.3: The converse of the Proposition 3.2 need not be true as shown in Example 3.6.

Example 3.6: From Example 3.5, the function F is advanced binary pre-continuous but not advanced binary continuous.

Proposition 3.3: Every advanced binary continuous function on an advanced binary topological space is advanced binary α-continuous.

Proof: Let (A_1, B_1) be an advanced binary open set in $(A, B, \mathcal{T}_{A})$. Since $F: (Z, \mathcal{T}) \rightarrow (A, B, \mathcal{T}_{A})$ is an advanced binary continuous function, it follows that $F^{-1}(A_1, B_1)$ is an advanced open set in $(Z, \mathcal{T})$. It is known that every advanced open set is an advanced α-open set. Therefore, $F^{-1}(A_1, B_1)$ is an advanced α-open set in $(Z, \mathcal{T})$. Hence, the inverse image of every advanced binary open set in $(A, B, \mathcal{T}_{A})$ is an advanced α-open set in $(Z, \mathcal{T})$. Thus, F is an advanced binary α-continuous.

Remark 3.4: The converse of the Proposition 3.3 need not be true as shown in Example 3.7.

Example 3.7: From Example 3.5, the function F is advanced binary α-continuous but not advanced binary continuous.

Proposition 3.4: Every advanced binary continuous function on an advanced binary topological space is advanced binary β -continuous.

Proof: Let (A_1, B_1) be an advanced binary open set in $(A, B, \mathcal{T}_{A})$ Since $F : (Z, \mathcal{T}) \rightarrow (A, B, \mathcal{T}_{A})$ is an advanced binary continuous function, it follows that $F^{-1}(A_1, B_1)$ is an advanced open set in $(Z, \mathcal{T})$. It is known that every advanced open set is an advanced β -open set. Therefore, $F^{-1}(A_1, B_1)$ is an advanced β -open set in $(Z, \mathcal{T})$. Hence, the inverse image of every advanced binary open set in $(A, B, \mathcal{T}_{A})$ is an advanced β -open set in $(Z, \mathcal{T})$. Thus, F is an advanced binary β –continuous.

Remark 3.5: The converse of the Proposition 3.4 need not be true as shown in Example 3.8.

Example 3.8: From Example 3.5, the function F is advanced binary β - continuous but not advanced binary continuous.

Remark 3.6: The concepts of advanced binary semi-continuous and advanced binary pre-continuous in advanced binary topology are independent of each other.

Proposition 3.5: Every advanced binary **α**-continuous in an advanced binary topological

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space is an advanced binary semi-continuous.

Proof: Let $\mathbf{F}: (\mathbf{Z}, \mathcal{T}) \rightarrow (\mathbf{A}, \mathbf{B}, \mathcal{T}_{\mathbb{A}})$ be an advanced binary $\boldsymbol{\alpha}$-continuous function. Let $(\mathbf{A_1}, \mathbf{B_1})$ be any advanced binary open set in $(\mathbf{A}, \mathbf{B}, \mathcal{T}_{\mathbb{A}})$. Since $\mathbf{F}$ is advanced binary $\boldsymbol{\alpha}$-continuous, the inverse image $\mathbf{F^{-1}}(\mathbf{A_1}, \mathbf{B_1})$ is an advanced $\boldsymbol{\alpha}$-open set in $(\mathbf{Z}, \mathcal{T})$. It is known that every advanced $\boldsymbol{\alpha}$-open set is advanced semi-open. Therefore, $\mathbf{F^{-1}}(\mathbf{A_1}, \mathbf{B_1})$ is an advanced semi-open set in $(\mathbf{Z}, \mathcal{T})$. Hence, the inverse image of every advanced binary open set in $(\mathbf{A}, \mathbf{B}, \mathcal{T}_{\mathbb{A}})$ is an advanced semi-open set in $(\mathbf{Z}, \mathcal{T})$. Hence, $\mathbf{F}$ is an advanced binary semi-continuous.

Remark 3.7: The converse of the Proposition 3.5 need not be true as shown in Example 3.8.

Example 3.9: Let $Z = \{a, b, c, d\}$, $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Then $T = \{\emptyset, \{b\}, \{b, c\}, \{c, d\}, \{b, c, d\}, Z\}$ and $\mathcal{T}_{\mathbb{A}} = \{(\emptyset, \emptyset), (\{a_1\}, \{b_1\}), (\{a_1\}, B), (\{a_2\}, B), (A, B)\}$. Clearly T is an advanced topology on Z and $\mathcal{T}_{\mathbb{A}}$ is advanced binary topology from A to B. Define $F: Z \rightarrow A \times B$ by $F(a) = F(b) = (a_1, b_1), F(c) = F(d) = (a_2, B)$, Now $F^{-1}(\emptyset, \emptyset) = \emptyset$, $F^{-1}(\{a_1\}, \{b_1\}) = \{a, b\}$, $F^{-1}(\{a_2\}, B) = \{c, d\}$, and $F^{-1}(A, B) = Z$, This shows that the inverse image of every advanced binary open set in $(A, B, \mathcal{T}_{\mathbb{A}})$ is advanced semi-open in (Z, T). Hence F is advanced binary semi-continuous but not advanced binary α-continuous because $\{a, b\}$ is advanced semi-open but not advanced α-open in (Z, T).

Proposition 3.6: Every advanced binary $\boldsymbol{\alpha}$-continuous in an advanced binary topological space is an advanced binary pre-continuous.

Proof: Let $\mathbf{F}: (\mathbf{Z}, \mathcal{T}) \rightarrow (\mathbf{A}, \mathbf{B}, \mathcal{T}_{\mathbb{A}})$ be an advanced binary $\boldsymbol{\alpha}$-continuous function. Let $(\mathbf{A_1}, \mathbf{B_1})$ be any advanced binary open set in $(\mathbf{A}, \mathbf{B}, \mathcal{T}_{\mathbb{A}})$. Since $\mathbf{F}$ is advanced binary $\boldsymbol{\alpha}$-continuous, the inverse image $\mathbf{F^{-1}}(\mathbf{A_1}, \mathbf{B_1})$ is an advanced $\boldsymbol{\alpha}$-open set in $(\mathbf{Z}, \mathcal{T})$. It is known that every advanced $\boldsymbol{\alpha}$-open set is advanced pre-open. Therefore, $\mathbf{F^{-1}}(\mathbf{A_1}, \mathbf{B_1})$ is an advanced pre-open set in $(\mathbf{Z}, \mathcal{T})$. Hence, the inverse image of every advanced binary open set in $(\mathbf{A}, \mathbf{B}, \mathcal{T}_{\mathbb{A}})$ is an advanced pre-open set in $(\mathbf{Z}, \mathcal{T})$. Thus, $\mathbf{F}$ is an advanced binary pre-continuous

Remark 3.8: The converse of the Proposition 3.6 need not be true as shown in Example 3.10.

Example 3.10: Let $Z = \{a, b, c, d\}$, $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Then $T = \{\emptyset, \{a, b\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}, Z\}$ and $\mathcal{T}_{\mathbb{A}} = \{(\emptyset, \emptyset), (\{a_1\}, \{b_1\}), (\{a_1\}, B), (\{a_2\}, B), (A, B)\}$. Clearly T is an advanced topology on Z and $\mathcal{T}_{\mathbb{A}}$ is advanced binary topology from A to B. Define $F: Z \rightarrow A \times B$ by $F(a) = F(c) = F(d) = (a_1, b_1), F(b) = (a_2, B)$, Now $F^{-1}(\emptyset, \emptyset) = \emptyset$, $F^{-1}(\{a_1\}, \{b_1\}) = \{a, c, d\}$, $F^{-1}(\{a_2\}, B) =$

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$\{b\}$, and $F^{-1}(A, B) = Z$, This shows that the inverse image of every advanced binary open set in $(A, B, \mathcal{T}_A)$ is advanced pre-open in (Z, T). Hence F is advanced binary pre-continuous but not advanced binary α-continuous because $\{a, c, d\}$ is advanced pre-open but not advanced α-open in (Z, T).

Proposition 3.7: Every advanced binary semi-continuous in an advanced binary topological space is an advanced binary **β**-continuous.

Proof: Let $\mathbf{F}: (\mathbf{Z}, \mathcal{T}) \to (\mathbf{A}, \mathbf{B}, \mathcal{T}_A)$ be an advanced binary semi-continuous function. Let $(\mathbf{A_1}, \mathbf{B_1})$ be any advanced binary open set in $(\mathbf{A}, \mathbf{B}, \mathcal{T}_A)$. Since $\mathbf{F}$ is advanced binary semi-continuous, the inverse image $\mathbf{F^{-1}}(\mathbf{A_1}, \mathbf{B_1})$ is an advanced semi-open set in $(\mathbf{Z}, \mathcal{T})$. It is known that every advanced semi-open set is advanced **β**-open. Therefore, $\mathbf{F^{-1}}(\mathbf{A_1}, \mathbf{B_1})$ is an advanced **β**-open set in $(\mathbf{Z}, \mathcal{T})$. Hence, the inverse image of every advanced binary open set in $(\mathbf{A}, \mathbf{B}, \mathcal{T}_A)$ is an advanced **β**-open set in $(\mathbf{Z}, \mathcal{T})$. Thus, $\mathbf{F}$ is an advanced binary **β**-continuous.

Remark 3.9: The converse of the Proposition 3.7 need not be true as shown in Example 3.11.

Example 3.11: Let $Z = \{a, b, c, d\}$, $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Then $T = \{\emptyset, \{b\}, \{c\}, \{b, c\}, Z\}$ and $\mathcal{T}_A = \{(\emptyset, \emptyset), (\{a_1\}, \{b_1\}), (\{a_2\}, B), (A, B)\}$. Clearly T is an advanced topology on Z and $\mathcal{T}_A$ is advanced binary topology from A to B. Define $F: Z \to A \times B$ by $F(a) = (a_1, b_1), F(b) = F(c) = F(d) = (a_2, B)$, Now $F^{-1}(\emptyset, \emptyset) = \emptyset$, $F^{-1}(\{a_1\}, \{b_1\}) = \{a\}$, $F^{-1}(\{a_2\}, B) = \{b, c, d\}$, and $F^{-1}(A, B) = Z$, This shows that the inverse image of every advanced binary open set in $(A, B, \mathcal{T}_A)$ is advanced β-open in (Z, T). Hence F is advanced binary β -continuous but not advanced binary semi-continuous because $\{b, c, d\}$ is advanced β-open but not advanced semi-open in (Z, T).

Proposition 3.8: Every advanced binary pre-continuous in an advanced binary topological space is an advanced binary **β**-continuous.

Proof: Let $\mathbf{F}: (\mathbf{Z}, \mathcal{T}) \to (\mathbf{A}, \mathbf{B}, \mathcal{T}_A)$ be an advanced binary pre-continuous function. Let $(\mathbf{A_1}, \mathbf{B_1})$ be any advanced binary open set in $(\mathbf{A}, \mathbf{B}, \mathcal{T}_A)$. Since $\mathbf{F}$ is advanced binary pre-continuous, the inverse image $\mathbf{F^{-1}}(\mathbf{A_1}, \mathbf{B_1})$ is an advanced pre-open set in $(\mathbf{Z}, \mathcal{T})$. It is known that every advanced pre-open set is advanced **β**-open. Therefore, $\mathbf{F^{-1}}(\mathbf{A_1}, \mathbf{B_1})$ is an advanced **β**-open set in $(\mathbf{Z}, \mathcal{T})$. Hence, the inverse image of every advanced binary open set in $(\mathbf{A}, \mathbf{B}, \mathcal{T}_A)$ is an advanced **β**-open set in $(\mathbf{Z}, \mathcal{T})$. Thus, $\mathbf{F}$ is an advanced binary **β**-continuous.

Remark 3.10: The converse of the Proposition 3.8 need not be true as shown in Example

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3.12.

Example 3.12: Let $Z = \{a, b, c, d\}$, $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Then $T = \{\emptyset, \{b\}, \{c\}, \{b, c\}, Z\}$ and $\mathcal{T}_A = \{(\emptyset, \emptyset), (\{a_1\}, \{b_1\}), (\{a_2\}, B), (A, B)\}$. Clearly T is an advanced topology on Z and $\mathcal{T}_A$ is advanced binary topology from A to B. Define $F: Z \to A \times B$ by $F(a) = (a_1, b_1), F(b) = F(c) = F(d) = (a_2, B)$, Now $F^{-1}(\emptyset, \emptyset) = \emptyset$, $F^{-1}(\{a_1\}, \{b_1\}) = \{a\}$, $F^{-1}(\{a_2\}, B) = \{b, c, d\}$, and $F^{-1}(A, B) = Z$, This shows that the inverse image of every advanced binary open set in $(A, B, \mathcal{T}_A)$ is advanced β-open in (Z, T). Hence F is advanced binary β -continuous but not advanced binary pre-continuous because $\{b, c, d\}$ is advanced β-open but not advanced pre-open in (Z, T).

Proposition 3.9: Every advanced binary α--continuous in an advanced binary topological space is an advanced binary **β**-continuous.

Proof: Let $\mathbf{F}: (\mathbf{Z}, \mathcal{T}) \to (\mathbf{A}, \mathbf{B}, \mathcal{T}_A)$ be an advanced binary α--continuous function. Let $(\mathbf{A_1}, \mathbf{B_1})$ be any advanced binary open set in $(\mathbf{A}, \mathbf{B}, \mathcal{T}_A)$. Since $\mathbf{F}$ is advanced binary α--continuous, the inverse image $\mathbf{F^{-1}}(\mathbf{A_1}, \mathbf{B_1})$ is an advanced α--open set in $(\mathbf{Z}, \mathcal{T})$. It is known that every advanced α--open set is advanced **β**-open. Therefore, $\mathbf{F^{-1}}(\mathbf{A_1}, \mathbf{B_1})$ is an advanced **β**-open set in $(\mathbf{Z}, \mathcal{T})$. Hence, the inverse image of every advanced binary open set in $(\mathbf{A}, \mathbf{B}, \mathcal{T}_A)$ is an advanced **β**-open set in $(\mathbf{Z}, \mathcal{T})$. Thus, $\mathbf{F}$ is an advanced binary **β**-continuous

Remark 3.11: The converse of the Proposition 3.9 need not be true as shown in Example 3.13.

Example 3.13: From Example 3.12, the function F is advanced binary β - continuous but not advanced binary α- continuous.

4. CONCLUSION

In this paper, the concept of advanced binary continuity has been introduced within the framework of advanced binary topological spaces. Several types of advanced binary continuous mappings, including semi, pre, α, and β-continuity, have been systematically studied. The relationships among these classes have been established and analyzed. Appropriate examples have been provided to illustrate their independence and hierarchical structure. These results contribute to the development of advanced binary topology and suggest directions for further research.

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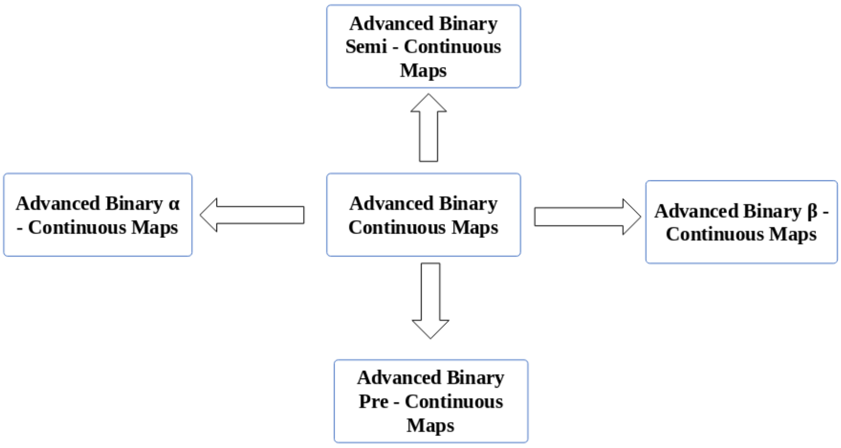

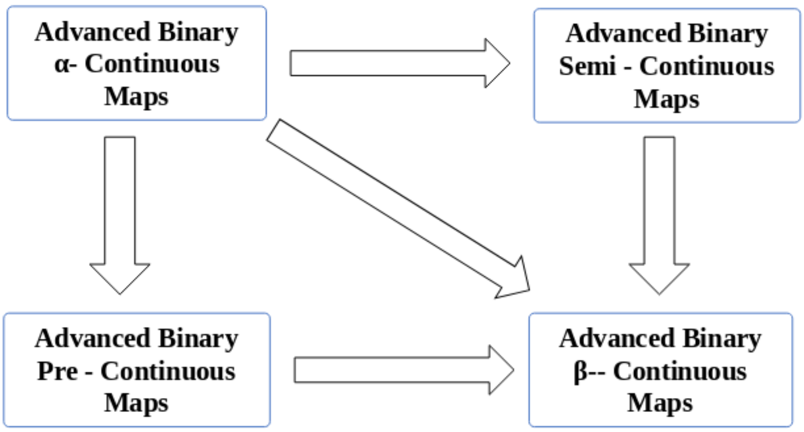

Figure: 1: Relationship of Advanced Continuous Maps

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