q$\mathcal{I}$- CONNECTEDNESS IN IDEAL BITOPOLOGICAL SPACES

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ABSTRACT

The purpose of this paper is to introduce and study the notion of q$\mathcal{I}$- connectedness in ideal bitopological spaces. We shall also study the notions of q$\mathcal{I}$- separated sets in ideal bitopological spaces.

Keywords*Ideal bitopological spaces, q$\mathcal{I}$- connected, q$\mathcal{I}$- separated sets, q$\mathcal{I}$- s-connected.*

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I.INTRODUCTION AND PRELIMINARIES

In 1961 Kelly introduced the concept of bitopological spaces as an extension of topological spaces [1]. A bitopological space (X, $\boldsymbol{\tau_1}$, $\boldsymbol{\tau_2}$)is a nonempty set X equipped with two topologies $\boldsymbol{\tau_1}$ and $\boldsymbol{\tau_2}$[1]. The study of quasi open sets in bitopological spaces was initiated by Datta in 1971 [2]. In a bitopological space (X, $\boldsymbol{\tau_1}$, $\boldsymbol{\tau_2}$) a set A of X is said to be quasi open if it is a union of a τ_1-open set and a τ_2-open set [2]**.** Complement of a quasi open set is termed quasi closed. Everyτ_1-open (resp. τ_2-open) set is quasi open but the converse may not be true. Anyunion of quasi open sets of X is quasi open in X. The intersection of all quasi closed sets which contains A is called quasi closure of A. It is denoted by qCl(A) [2]. The union of quasi open subsets of A is called quasi interior of A. It is denoted by qInt(A)[2].

The study of ideal topological spaces was initiated by Kuratowski[3] and Vaidyanathaswamy [4]. Applications to various fields were further investigated by Dontchev [5], Hatir, Keskin and Noiri [6], Jankovic and Hamlett [7], Nasef and Mahmoud [8], Navaneethakrishnan and Joseph [9] and others.

An Ideal $\mathcal{I}$ on a topological space (X, $\boldsymbol{\tau}$) is a non empty collection of subsets of X which satisfies:

i. $A \in \mathcal{I}$and $B \subset A \Rightarrow B \in \mathcal{I}$ and

ii. $A \in \mathcal{I}$and $B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$

An ideal topological space is a topological space (X, $\boldsymbol{\tau}$) with an ideal $\mathcal{I}$ on X, and is denoted by (X, $\boldsymbol{\tau}$, $\mathcal{I}$). If $\mathcal{P}$(X) is the set of all subsets of X, in a topological space (X, $\boldsymbol{\tau}$) a set operator $(.)^*$:$\mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is called the local mapping [5] of A with respect to τ and $\mathcal{I}$ and is defined as follows: $A^*(\tau, \mathcal{I}) = \{x \in X \mid U \cap A \notin \mathcal{I}, \forall\ U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau \mid x \in U\}$.

Definition 1.1.[10].If (X, τ_1, τ_2) is a bitopological space then $(X, \tau_1, \tau_2, \mathcal{I})$ is an ideal bitopological space if $\mathcal{I}$ is an ideal on X.

Given an ideal bitopological space $(X,\tau_1,\tau_2,\mathcal{I})$ the quasi local mappingof A with respect to τ_1, τ_2 and $\mathcal{I}$ denoted by A_q^* $(\tau_1,\tau_2,\mathcal{I})$ (in shortA_q^*) is defined as follows: $A_q^*(\tau_1,\tau_2,\mathcal{I}) = \{x \in X \mid U \cap A \notin \mathcal{I}, \forall$ quasi open set U containing x$\}$[11].

Definition1.2. [11].A subset A of an ideal bitopological space $(X,\tau_1,\tau_2,\mathcal{I})$ is q$\mathcal{I}$- open if $A \subset$qInt(A_q^*) and q$\mathcal{I}$-closed if its complement is q$\mathcal{I}$- open.

Definition1.3.[11].A mapping f: $(X,\tau_1,\tau_2,\mathcal{I}) \to (Y,\sigma_1,\sigma_2)$ is called a q$\mathcal{I}$- continuous if $f^{-1}(V)$ is a q$\mathcal{I}$- open set in X for every quasi open set V of Y.

Definition1.4. [11].In an ideal bitopological space $(X,\tau_1,\tau_2,\mathcal{I})$ the quasi $*$-closure of A of X denoted by $qCl^*(A)$ is defined by $qCl^*(A) = A \cup A_q^*$.

Definition1.5. [11].A subset A of an ideal bitopological space $(X,\tau_1,\tau_2,\mathcal{I})$ is said to be a q$\mathcal{I}$- neighbourhood of a point $x \in X$ if $\exists$ a q$\mathcal{I}$- open set O such that $x \in O \subset A$.

Definition1.6. [11].Let A be a subset of an ideal bitopological space $(X,\tau_1,\tau_2,\mathcal{I})$ and $x \in X$. Then x is called a q$\mathcal{I}$-interior point of A if $\exists$ V a q$\mathcal{I}$- open set in X such that $x \in V \subset A$.

The set of all q$\mathcal{I}$- interior points of Ais called the q$\mathcal{I}$- interior of A and is denoted by q$\mathcal{I}$Int(A).

Definition1.7.[11].Let A be a subset of an ideal bitopological space $(X,\tau_1,\tau_2,\mathcal{I})$ and $x \in X$. Then x is called a q$\mathcal{I}$-cluster point of A, if $V \cap A \neq \emptyset$. for every q$\mathcal{I}$- open set V in X. The set of all q$\mathcal{I}$-cluster points of A denoted by q$\mathcal{I}$Cl(A) is called the q$\mathcal{I}$-closure of A .

Definition 1.8. [12].An ideal topological space $(X, \tau, \mathcal{I})$ is called *-connected if X cannot be written as the disjoint union of a nonempty open set and a nonempty *-open set.

Definition 1.9.[13].An ideal bitopological space $(X, \tau_1, \tau_{2,} \mathcal{I})$ is called pairwise *-connected ifX cannot be written as the disjoint union of a nonempty τ_i open set and a nonempty τ_j^* -open set. $\{i , j = 1, 2;\ i \neq j\}$

Definition 1.10.[13].Nonempty subsets A, B of an ideal bitopological space $(X, \tau_1, \tau_{2,} \mathcal{I})$, are called pairwise *-separated if $\tau_i Cl^*(A) \cap B = A \cap \tau_j Cl(B) = \phi$. $\{i , j = 1 , 2;\ i \neq j\}$

II. q$\mathcal{I}$-CONNECTEDNESS IN IDEAL BITOPOLOGICAL SPACES

Definition 2.1.An ideal topological space $(X, \tau_1, \tau_{2,} \mathcal{I})$ is called q$\mathcal{I}$- connected if X cannot be written as the disjoint union of a nonempty quasi open set and a nonempty q$\mathcal{I}$- open set.

Definition 2.2.Nonempty subset A, B of an idealbitopological space $(X, \tau_1, \tau_{2,} \mathcal{I})$ are called q$\mathcal{I}$-separated if $qCl(A) \cap B = A \cap q\mathcal{I}Cl(B) = \phi$.

Theorem 2.1.If A, B are q$\mathcal{I}$-separated sets of an ideal bitopological space $(X, \tau_1, \tau_{2,} \mathcal{I})$ and $A \cup B \in \tau_1 \cap \tau_2$ then A isq$\mathcal{I}$-openand B is quasiopen.

Proof: Since A and B are q$\mathcal{I}$-separated in X, then $B = (A \cup B) \cap (X - qCl(A))$. Since $A \cup B$ is biopenand $qCl(A)$ isquasi closed in X, B is quasiopen in X.Similarly$A = (A \cup B) \cap (X - q\mathcal{I}Cl(B))$ and we obtain that A is q$\mathcal{I}$-open in X.

Theorem 2.2.Let $(X,\tau_1,\tau_{2,}\mathcal{I})$ be an ideal bitopological space and $A, B \subset Y \subset X$. Then A and Bareq$\mathcal{I}$-separated in Y if and only if A, B are q$\mathcal{I}$-separated in X.

Proof: It follows from$qCl(A) \cap B = A \cap q\mathcal{I}Cl(B) = \phi$ and the fact that $A, B \subset Y \subset X$.

Theorem 2.3.Iff: $(X,\tau_1,\tau_{2,}\mathcal{I}) \rightarrow (Y, \sigma_{1,} \sigma_2)$ is a q$\mathcal{I}$-continuous onto mapping. Then if $(X, \sigma_{1,} \sigma_{2,} \mathcal{I})$ is a q$\mathcal{I}$-connected ideal bitopological space $(Y, \sigma_{1,} \sigma_2)$ is also quasi connected.

Proof:It is known that connectedness is preserved by continuous surjections. Hence everyq$\mathcal{I}$-open set is also quasi open. Hence, q$\mathcal{I}$-connected space is also quasi connected.

Definition 2.3.A subset A of an ideal bitopological space $(X,\tau_1,\tau_{2,}\mathcal{I})$is calledq$\mathcal{I}$-*s*-connected if A is not the union of two nonempty q$\mathcal{I}$-separated sets in $(X,\tau_1,\tau_{2,}\mathcal{I})$.

Theorem 2.4.Let Y be a biopen subset of an ideal bitopological space $(X,\tau_1,\tau_{2,}\mathcal{I})$

The following are equivalent:

i. Y is q$\mathcal{I}$-s-connected in (X, τ_1, τ_2, $\mathcal{I}$)

ii. Y is q$\mathcal{I}$- connected in (X,τ_1,τ_2,$\mathcal{I}$).

Proof:i)⇒ii) Let Y be q$\mathcal{I}$-s-connected in (X, τ_1, τ_2, $\mathcal{I}$) and suppose that Y is not q$\mathcal{I}$-connected in (X, τ_1, τ_2, $\mathcal{I}$). There exist non empty disjoint quasi open set A,in Y andq$\mathcal{I}$- open setB in Y s.t Y = A ∪ B. Since Y is biopen in X and A and B are quasi open and q$\mathcal{I}$- open in Xrespectively andA and B are disjoint, then qCl(A) ∩ B = ∅ = A ∩q$\mathcal{I}$Cl(B). This implies that A, B are q$\mathcal{I}$-separated sets in X. Thus, Y is not q$\mathcal{I}$-s-connected in (X, τ_1, τ_2, $\mathcal{I}$). Hence we arrive at a contradiction and Y is q$\mathcal{I}$- connected in (X, τ_1 ,τ_2, $\mathcal{I}$).

ii)⇒i) Suppose Y is q$\mathcal{I}$-connected in (X, τ_1, τ_2, $\mathcal{I}$) and Y is not q$\mathcal{I}$-s-connected in(X,τ_1,τ_2,$\mathcal{I}$). There exist two q$\mathcal{I}$-separated sets A, Bs.tY = A∪ B. By Theorem 2.1,A andB are q$\mathcal{I}$-open andquasi open inY respectively. Since Y is biopen in X, obviouslyA and B are q$\mathcal{I}$-open and quasiopen in X respectively. Also Y is q$\mathcal{I}$-connected so Y cannot be written as the disjoint union of a nonempty quasi open set and a nonempty q$\mathcal{I}$- open set. This is a contradiction and Y is q$\mathcal{I}$-s-connected.

Theorem 2.5.Let (X,τ_1,τ_2,$\mathcal{I}$) be an ideal bitopological space. If A is a q$\mathcal{I}$-s-connected set of X and H, G are q$\mathcal{I}$-separated sets of X with A ⊂ H ∪ G, then either A⊂ H or A ⊂ G.

Proof: Let A⊂ H ∪ G. Since A = (A∩ H) ∪(A∩ G), then (A∩ G) ∩qCl(A ∩H) ⊂ G ∩q$\mathcal{I}$Cl(H) = ∅. By similar reasoning, we have (A ∩ H) ∩qCl(A ∩ G) ⊂ H ∩q$\mathcal{I}$Cl(G) = ∅. If A ∩ H and A ∩ G are nonempty, then A is not q$\mathcal{I}$-s-connected. This is a contradiction. Thus, either A ∩H = ∅or A ∩G = ∅This implies that either A ⊂ H or A ⊂ G.

Theorem 2.6.If A is a q$\mathcal{I}$-s-connected set of an ideal bitopologicalspace (X, τ_1, τ_2, $\mathcal{I}$)and A ⊂B ⊂qCl (A)∩q$\mathcal{I}$Cl(B) then B is q$\mathcal{I}$-s-connected.

Proof:The theorem can easily be proved by taking the contradiction.

Theorem 2.7.If {M_i: i ∈ I} is a nonempty family of q$\mathcal{I}$-s-connected sets of an ideal bitopological space(X, τ_1, τ_2, $\mathcal{I}$)with $\bigcap_{i\in I} Mi \neq \phi$ Then $\bigcup_{i\in I} Mi$ is q$\mathcal{I}$-s-connected.

Proof:Suppose that $\bigcup_{i\in I} Mi$is not q$\mathcal{I}$-s-connected. Then we have $\bigcup_{i\in I} Mi$= H ∪ G,where H and G are q$\mathcal{I}$-separated sets in X. Since $\bigcap_{i\in I} Mi \neq \phi$we have a point x in $\bigcap_{i\in I} Mi$. Since x ∈$\bigcup_{i\in I} Mi$, either x ε H or x ε G. Suppose that x ε H. Since x ε $M_i$for each i ε I, then $M_i$and H intersect for each i ε I. By theorem2.5: M_i⊂ H or M_i⊂ G. Since H and G are disjoint, M_i⊂ H for all i ε I and hence$\bigcup_{i\in I} Mi$⊂ H. This implies that G is empty. This is a contradiction.

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Suppose that x ε G. By similar way, we have that H is empty which is a contradiction. Thus, $\bigcup_{i\in I} Mi$is q$\mathcal{I}$-s-connected.

Theorem2.8.Suppose that $\{M_n$:n ε N$\}$ is an infinite sequence ofq$\mathcal{I}$-connected open sets of an ideal space (X, τ_1, τ_2, $\mathcal{I}$)and $M_n \cap M_{n+1} \neq \phi$ for each n ε N. Then $\bigcup_{i\in I} Mi$ is q$\mathcal{I}$-s-connected.

Proof:By induction and Theorems 2.4 and 2.7, the set $P_n = \bigcup_{k\leq n} Mk$is a q$\mathcal{I}$-connected open set for each n ε N. Also, $P_n$has a nonempty intersection. Thus $\bigcup_{n\in N} Mn$ is q$\mathcal{I}$-connected.

Definition 2.4.Let X be anideal bitopological space (X, τ_1, τ_2, $\mathcal{I}$)and x $\in$ X. The union of all q$\mathcal{I}$-s-connected subsets of X containing x is called the q$\mathcal{I}$-component of X containing x.

Theorem 2.9.Each q$\mathcal{I}$-component of an ideal bitopological space (X, τ_1, τ_2, $\mathcal{I}$)is a maximal q$\mathcal{I}$-sconnected set of X.

Proof: Obvious.

Theorem 2.10.The set of all distinctq$\mathcal{I}$-components of an ideal bitopological space (X, τ_1, τ_2, $\mathcal{I}$) forms a partition of X.

Proof: Let A and B be two distinct q$\mathcal{I}$-components of an ideal bitopological space (X, τ_1, τ_2, $\mathcal{I}$) containing x and y respectively $\{x \neq y\}$. Suppose that A and B intersect. Then, by Theorem 2.7, A $\cup$B is q$\mathcal{I}$-s-connected in X. Also, A, B$\subseteq$ A $\cup$B,so A, Bare not maximal and thus A, B are disjoint. Hence they partition X. by induction it can easily be proved that the set of all distinct q$\mathcal{I}$-components of X forms a partition of X.

Theorem 2.11.Each q$\mathcal{I}$- component of an ideal bitopological space (X, τ_1, τ_2, $\mathcal{I}$) is q$\mathcal{I}$-closed in X.

Proof: Let A be a q$\mathcal{I}$-component of X. Therefore qCl(A) is q$\mathcal{I}$-s-connected and A = qCl (A). Thus, A is q$\mathcal{I}$-closed in X.

III.CONCLUSION

Ideal Bitopological Spaces is an extension for both Ideal Topological Spaces and Bitopological Spaces. It has opened new areas of research in Topology and in the study of topological concepts via Fuzzy ideals in Ideal Bitopological Spaces. The application of the results obtained would be remarkable in other branches of science too.

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