International Conference on Recent Innovation in Engineering, Science and Management (RIESM-19)

Osmania University Centre for International Program, Osmania University Campus, Hyderabad (India)

23rd November 2019 www.conferenceworld.in ISBN : 978-81-943584-3-5

IMPORTANCE OF LYAPUNOV FUNCTIONS

K KAMALA PRIYADARSHINI
GUEST FACULTY , DEPARTMENT OF MATHEMATICS
RGUKT IIIT, NUZVID CAMPUS, NUZVID 521 202
KRISHNA DT. A.P

Abstract

We construct strict Lyapunov functions for broad classes of nonlinear systems satisfying Matrosov type conditions. Our new constructions are simpler than the designs available in the literature. We illustrate the practical interest of our designs using a globally asymptotically stable biological model.

Key Words: application, construction, essential, functions, Lyapunov, nonlinear, quadratic, role

Introduction

Lyapunov functions play an essential role in modern nonlinear systems analysis and controller design. Oftentimes, *non*-strict Lyapunov functions are readily available. However, *strict* (i.e., *strong*) Lyapunov functions are preferable since they can be used to quantify the effects of disturbances; see the precise definitions below. Strict Lyapunov functions have been used in several biological contexts e.g. to quantify the effects of actuator noise and other uncertainty on the steady state concentrations of competing species in chemostats [13], but their explicit construction can be challenging. For some large classes of systems, there are mechanisms for transforming non-strict Lyapunov functions into the required strict Lyapunov functions e.g. [4], [11], [12], [14], [15].

For systems satisfying conditions of Matrosov's type [8], [10], strict Lyapunov functions were constructed in [15], under very general conditions. However, the generality of the assumptions in[15] makes its constructions complicated and therefore difficult to apply. Moreover, the Lyapunov functions provided by [15] are not locally bounded from below by positive definite quadratic functions, even for asymptotically stable linear systems, which admit a quadratic strict Lyapunov function. The shape of Lyapunov functions, their local properties and their simplicity matter when they are used to investigate robustness and construct feedbacks and gains.

In the present work, we revisit the problem of constructing strict Lypaunov functions under Matrosov's conditions. Our results have the following desirable features. First, they lead to simplified constructions of strict Lyapunov functions; see Remark 3 below. For a large family of systems, the Lyapunov functions we construct have the added advantage of being locally bounded from below by positive definite quadratic functions, with time derivatives along the trajectories that are locally bounded from above by negative definite quadratic functions. Second, our work does not require a non-strict positive definite radially unbounded Lyapunov function. Rather, we only require a non-strict positive definite function whose derivative along the trajectories is non-positive.

One of our motivations is that one can frequently find non-strict Lyapunov-like functions which are not proper but which make it possible to establish global asymptotic stability of an equilibrium point. For instance, the celebrated Lyapunov function from [5] for a multi-species chemostat (also reported in [16]) is not proper. In such cases, the stability proof is often based on the fact

ISBN : 978-81-943584-3-5

that the models are derived from mass balance properties [1] leading to the boundedness of the trajectories. Our work yields robustness in the sense of input-to-state stability (ISS). The ISS notion is a fundamental paradigm of nonlinear control that makes it possible to quantify the effects of uncertainty [17], [18]. While our assumptions are more restrictive than those used in [8], [15], they are general in the sense that, to the best of our knowledge, they are satisfied by all examples whose stability can be established by the generalized Matrosov's theorem; specifically, see e.g. the examples in [15] whose auxiliary functions satisfy our Assumptions 1-2 below.

Definitions and Notation

We omit the arguments of our functions when they are clear. We use the standard classes of comparison functions K_∞ and KL; see [18] for their well known definitions. We always assume that $D \subseteq R^n$ is an open set for which $0 \in D$. A function $V : D \times R \to R$ is *positive definite* on D provided $V(0,t) \equiv 0$ and $\inf_t V(x,t) > 0$ for all $x \in D \setminus \{0\}$. A function V is *negative definite* provided $-V$ is positive definite. Let $|\cdot|$ (resp., $|\cdot|_\infty$) denote the standard Euclidean norm (resp., essential supremum). We always assume that our functions are sufficiently smooth. Consider a general nonlinear system

$$\dot{x} = F(x, t, \delta(t))$$

evolving on a forward invariant open set G that is diffeomorphic to R^n, with disturbances δ in the set $L_\infty(C)$ of all measurable essentially bounded functions valued in a given subset C of Euclidean space. Assume $0 \in G$, $0 \in C$, $F \in C^1$ with $F(0,t,0) \equiv 0$, where C^1 means continuously differentiable.

A C^1 function $V : D \times R \to R$ is a *Lyapunov-like function* for (1) with $\delta \equiv 0$ provided V is positive definite and $\dot{V}(x, t) := \frac{\partial V}{\partial t}(x, t) + \frac{\partial V}{\partial x}(x, t)F(x, t, 0) \leq 0$ for all $x \in D$ and $t \geq 0$. If in addition $\dot{V}(x, t)$ is negative definite, then V is a *strict Lyapunov-like function* for (1) with $\delta \equiv 0$. A function $W : D \to R$ is *radially unbounded* (or *proper*) provided $\lim_{x\in D,|x|\to+\infty} W(x) = +\infty$. A (strict) Lyapunov-like function is a *(strict) Lyapunov function* provided it is also proper.

Let $t \mapsto \varphi(t; t_o, x_o, \delta)$ denote the solution for (1) with arbitrary initial condition $x(t_0) = x_o$ and any $\delta \in L_\infty(C)$, which we always assume to be uniquely defined on $[t_o, +\infty)$. Let $M(G)$ denote the set of all continuous functions $M : G \to [0, \infty)$ for which (A) $M(0) = 0$ and (B) $M(x) \to +\infty$ as $x \to$ boundary(G) or $|x| \to +\infty$ while remaining in G. We say that (1) is *ISS on* G *with disturbances in* C (or just ISS when G and C are clear) [17], [18] provided there exist $\beta \in KL$, $M \in M(G)$ and $\gamma \in K_\infty$ such that $|\varphi(t; t_o, x_o, \delta)| \leq \beta(M(x_o), t - t_o) + \gamma(|\delta|_\infty)$ for all $t \geq t_o \geq 0$, $x_o \in G$, and $\delta \in L_\infty(C)$. When $G = R^n$ and $M(x) = |x|$, this becomes the usual ISS definition. The ISS property reduces to the standard (uniformly) globally asymptotically stable condition when $\delta \equiv 0$ but is far more general because it quantifies the effects of disturbances

ISBN : 978-81-943584-3-5

Main Result

A. Statement of Assumptions and Result

For simplicity, we first state our main result for time invariant systems $\dot{x} = f(x)$ evolving on D; see Section IV for generalizations to time-varying systems $\dot{x} = f(x, t)$. We assume:

Assumption 1: There exist an integer $j \geq 2$; functions $V_i : D \to \mathbb{R}$, $N_i : D \to [0, +\infty)$, and $\varphi_i : [0, +\infty) \to (0, +\infty)$; and constants $a_i \in (0, 1]$ such that (a) $V_i(0) = N_i(0) = 0$ for all i; (b) $\nabla V_1(x) f(x) \leq -N_1(x)$ and $\nabla V_i(x) f(x) \leq -N_i(x) + \varphi_i(V_1(x)) \sum_{l=1}^{i-1} N_l^{a_i}(x) V_1^{1-a_i}(x)$ for $i = 2, \ldots, j$ for all $x \in D$, and (c) the function V_1 is positive definite on D.

Assumption 2: (i) There exists a function $\rho : [0, +\infty) \to (0, +\infty)$ such that $\sum_{l=1}^{j} N_l(x) \geq \rho(V_1(x)) V_1(x)$ for all $x \in D$. (ii) There exist functions $p_2, \ldots, p_j : [0, +\infty) \to [0, +\infty)$ such that $|V_i(x)| \leq p_i(V_1(x)) V_1(x)$ for all $x \in D$ holds for $i = 2, 3, \ldots, j$.

Theorem 1: Assume that there exist $j \geq 2$ and functions satisfying Assumptions 1-2. Then one can build explicit functions $k_l, \Omega_l \in \mathcal{K}_\infty \cap C^1$ such that $S(x) = \sum_{l=1}^{j} \Omega_l (k_l(V_1(x)) + V_l(x))$ satisfies

$$S(x) \geq V_1(x) \quad \text{and} \quad \nabla S(x) f(x) \leq -\frac{1}{4} \rho(V_1(x)) V_1(x) \quad (2)$$

for all $x \in D$.

Remark 1: The proof of Theorem 1 uses the triangular structure of the inequalities in Assumption 1(b) in an essential way. The differences between Assumptions 1-2 and the assumptions from [15] are these. First, while Assumption 1 above ensures that V_1 is positive definite but not necessarily proper, [15] requires a radially unbounded non-strict Lyapunov function. Second, our Assumption 1 is a restrictive version of [15, Assumption 2] because we specify the local properties of the functions which correspond to the χ_i's of [15, Assumption 2]. Finally, our Assumption 2 imposes relations between the functions N_i and V_1, which are not required in [15]. In Section IV, we extend our result to time-varying systems. Note that we do not require $V_2, \ldots, V_j$ to be nonnegative.

Remark 2: If $D = \mathbb{R}^n$ and V_1 is radially unbounded, then (2) implies that S is a strict Lyapunov function for $\dot{x} = f(x)$. If V_1 is not radially unbounded, then one cannot conclude from Lyapunov's theorem that the origin is globally asymptotically stable. However, in many cases, global asymptotic stability can be proved through a Lyapunov-like function and extra arguments. We illustrate this in Section V. If V_1 is bounded from below by a positive definite quadratic form in a neighborhood of 0, then we get positive definite quadratic lower bounds on S (by (2)) and $\frac{1}{4}\rho(V_1(x))V_1(x)$ near 0.

Extension to Time-Varying Systems

One can prove an analog of Theorem 1 for $\dot{x} = f(x,t)$, as follows. We assume that there exists $R \in \mathcal{K}_\infty$ such that $|f(x,t)| \leq R(|x|)$ everywhere, and that time-varying analogs of Assumptions 1-2 hold. These analogs of Assumptions 1-2 are obtained by replacing their arguments x by (x,t), and $\nabla V_i(x) f(x)$ by $\dot{V}_i(x,t) = \frac{\partial V_i}{\partial t}(x,t) + \frac{\partial V_i}{\partial x}(x,t) f(x,t)$, assuming $N_i(0,t) \equiv V_i(0,t) \equiv 0$. More generally, assume that these time-varying versions of Assumptions 1-2 hold *except* that the lower bound on $\Sigma_i N_i$ is replaced by a relation of the form

ISBN : 978-81-943584-3-5

$$S_j(x, t) := \sum_{l=1}^{j} N_l(x,t) \geq \underline{p}(t)\rho(V_1(x,t))V_1(x,t). \quad (6)$$

Here ρ is again positive, and $\underline{p}(t)$ is assumed to be non-negative and admit constants $\bar{B}, T, p_m > 0$ such that $\int_t^{T+t} \underline{p}(s)ds > p_m$ and $\underline{p}(t) \leq \bar{B}$ for all t. This allows $\underline{p}(t) = 0$ for some t's (e.g., $\underline{p}(t) = \cos^2(t)$ and $T = \pi$), so $S_j(x, t)$ is not necessarily positive definite. Nevertheless, we can prove an analog of Theorem 1 in this situation, as follows.

Set $V_{j+1}(x,t) = \int_{t-T}^{t} \int_{s}^{t} \underline{p}(l)\, dl\, ds\ V_1(x,t)$. Since $\dot{V}_1(x,t) \leq 0$ and $\underline{p}$ and V_1 are nonnegative,

$$\dot{V}_{j+1} = -V_1(x,t)\int_{t-T}^{t} \underline{p}(l)dl + T\underline{p}(t)V_1(x,t) + \dot{V}_1 \int_{t-T}^{t}\int_{s}^{t} \underline{p}(l)dlds$$

$$\leq -p_m V_1(x,t) + T\underline{p}(t)V_1(x,t) \leq -p_m V_1(x,t) + T\frac{S_j(x,t)}{\rho(V_1(x,t))}.$$

along the trajectories of $\dot{x} = f(x, t)$, where the first inequality is by our choice of p_m. Therefore,

$$\dot{V}_i \leq -N_i(x,t) + \varphi_i(V_1(x,t))\sum_{l=1}^{i-1} N_l(x,t)^{a_i} V_1(x,t)^{1-a_i} \quad \text{for } 2 \leq i \leq j, \text{ and}$$

$$\dot{V}_{j+1} \leq -N_{j+1}(x,t) + \varphi_{j+1}(V_1(x,t))\sum_{l=1}^{j} N_l(x,t)$$

with $N_{j+1}(x,t) = p_m V_1(x,t)$ and $\varphi_{j+1}(V_1(x,t)) = \frac{T}{\rho(V_1(x,t))}$. Also, $\sum_{l=1}^{j+1} N_l(x,t) \geq p_m V_1(x,t)$, $\dot{V}_1 \leq -N_1(x,t)$, and $|V_{j+1}(x,t)| \leq T^2\bar{B}V_1(x,t)$. Therefore, the properties required to apply the time-varying version of Theorem 1 are satisfied by $V_1, V_2, \ldots, V_{j+1}$.

A. *Robustness Result*

It is important to assess the robustness of a control design to bounded uncertainties before implementing the controller. Indeed, biological systems are known to have highly uncertain dynamics.

This is especially the case for waste water treatment processes made up of a complex mixture of bacteria. In [9], good performance of the controller was observed but could not be explained by a theoretical approach. Here we prove that an appropriate adaptive controller gives ISS of the relevant error dynamics to disturbances; see Section II for the definitions and motivations for ISS. We focus on the system (9) for cases where s_{in} is replaced by $H_{in}(t) = s_{in} + \delta_1(t)$ and, for an arbitrary positive constant $K > 0$, the adaptive control is given by

$$u = (\gamma + \delta_2(t))y_1, \qquad \dot{\gamma} = -Ky_1(\gamma - \gamma_m)(\gamma_M - \gamma)(\tilde{s} + \delta_3(t))$$

where the disturbances $\delta_1(t)$ and $\delta_3(t)$ are bounded in absolute value by a constant $\bar{\delta}_1$ and the disturbance $\delta_2(t)$ is bounded by a constant $\bar{\delta}_2$; we specify the $\bar{\delta}_i$'s below.

We maintain the assumptions and notation from the preceding subsections. We also assume

$$k/\lambda < (\gamma_m - \bar{\delta}_2)(s_{in} - \bar{\delta}_1), \quad \bar{\delta}_1 < s_{in}, \quad \text{and} \quad \bar{\delta}_2 < \gamma_m. \tag{20}$$

In particular, we keep the definitions of x_* and γ_* from (11) and the first sentence after unchanged; we do not replace s_{in} by $H_{in}(t)$ in the expressions for x_* and γ_*. Our analysis will use the function S from (17) extensively. To specify our bounds $\bar{\delta}$, we use the constants

$$\Xi = [\Upsilon_1 + 2\gamma_M^2/\nu_* + K(\gamma_M - \gamma_m)^2]/\gamma_m \quad \text{and} \quad \Upsilon_2 = \min\{\gamma_* - \gamma_m, \gamma_M - \gamma_*\}$$

where Υ_1 is from Section V-A. See Section V-C for an example with specific bounds $\bar{\delta}_i$.

Replacing s_{in} with H_{in} in (9), and using u from (19) and the expression for y_1, we get

$$\begin{cases} \dot{s} = y_1(\gamma + \delta_2(t))(s_{in} + \delta_1(t) - s) - \frac{k}{\lambda}\left[1 - \alpha(\gamma + \delta_2(t))x\right], \\ \dot{x} = y_1\left[1 - \alpha(\gamma + \delta_2(t))x\right], \\ \dot{\gamma} = -Ky_1(\gamma - \gamma_m)(\gamma_M - \gamma)(\tilde{s} + \delta_3(t)). \end{cases}$$

For simplicity, we restrict to the attractive and invariant domain where $0 < s < 2s_{in}$ which, in practice, is the domain of interest (but a result on the entire set where $s \geq 0$ can be proved). Using (20) and arguing as we did to obtain Lemma 2, the time re-scaling $\tau = \int_{t_0}^{t} y_1(l)\,dl$ yields

$$\begin{cases} \dot{\tilde{s}} = -\gamma\tilde{s} + \tilde{\gamma}\nu_* + \delta_2(t)(s_{in} - s) + (\gamma + \delta_2(t))\delta_1(t), \\ \dot{\tilde{x}} = -\alpha(\gamma + \delta_2(t))\tilde{x} - \alpha(\delta_2(t) + \tilde{\gamma})x_*, \\ \dot{\tilde{\gamma}} = -K(\gamma - \gamma_m)(\gamma_M - \gamma)\tilde{s} - K(\gamma - \gamma_m)(\gamma_M - \gamma)\delta_3(t) \end{cases}$$

Conclusion

We provided new strict Lyapunov function constructions for nonlinear systems that satisfy Ma- trosov's conditions. The advantages of our constructions lie in their simplicity and their applicability to the various examples whose stability can be established by the generalized Matrosov theorem. We demonstrated the efficacy of our methods through a class of biotechnological models with disturbances, which are of compelling engineering interest.

References

[1] G. Bastin and D. Dochain, *On-line Estimation and Adaptive Control of Bioreactors*. Amsterdam: Elsevier, 1990.print.

[2] O. Bernard, B. Chachuat, A. Hélias, and J. Rodriguez, "Can we assess the model complexity for a bioprocess? Theory and example of the anaerobic digestion process," *Water Science and Technology*, vol. 53, no. 1, pp. 85–92, 2006.print.

[3] O. Bernard, Z. Hadj-Sadok, D. Dochain, A. Genovesi, and J.-P. Steyer, "Dynamical model development and parameter identification for an anaerobic wastewater treatment process," *Biotechnology and Bioengineering*, vol. 75, no. 4, pp. 424– 438, 2001.print.

[4] L. Faubourg and J-B. Pomet, "Control Lyapunov functions for homogeneous "Jurdjevic-Quinn" systems," *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 5, pp. 293–311, 2000.print.

[5] S.B. Hsu, "Limiting behavior for competing species," *SIAM J. Appl. Math.*, vol. 34, no. 4, pp. 760–763, 1978.print.

[6] M. Krstic, I. Kanellakopoulos, and P. Kokotovic, *Nonlinear and Adaptive Control Design*. New York: Wiley, 1995.print.

[7] J.P. LaSalle, "Stability theory for ordinary differential equations," *J. Diff. Eq.*, vol. 4, pp. 57–65, 1968.print.

[8] A. Loria, E. Panteley, D. Popovic´ and A. Teel, "A nested Matrosov theorem and persistency of excitation for uniform convergence in stable non-autonomous systems," *IEEE Trans. Autom. Control*, vol. 50, no. 2, pp. 183–198, Feb. 2005.print.

[9] L. Mailleret, O. Bernard, and J-P. Steyer, "Nonlinear adaptive control for bioreactors with unknown kinetics," *Automatica*, vol. 40, no. 8, pp. 1379–1385, Aug. 2004.print.

[10] V.M. Matrosov, "On the theory of stability of motion," *J. Appl. Math. Mech.*, vol. 26, no. 6, pp. 1560–1572, 1962.print.

[11] F. Mazenc and M. Malisoff, "Further constructions of control-Lyapunov functions and stabilizing feedbacks for systems satisfying the Jurdjevic-Quinn conditions," *IEEE Trans. Autom. Control*, vol. 51, no. 2, pp. 360–365, Feb. 2006.print.

[12] F. Mazenc and M. Malisoff, "Lyapunov function constructions for slowly time-varying systems," *Math. Control Signals Syst.*, vol. 19, no. 1, pp. 1–21, 2007.print.

[13] F. Mazenc, M. Malisoff, and P. De Leenheer, "On the stability of periodic solutions in the perturbed chemostat," *Math. Biosci. Eng.*, vol. 4, no. 2, pp. 319–338, 2007.print.

[14] F. Mazenc and D. Nesic, "Strong Lyapunov functions for systems satisfying the conditions of LaSalle," *IEEE Trans. Autom. Control*, vol. 49, no. 6, pp. 1026–1030, June 2004.print.

[15] F. Mazenc and D. Nesic, "Lyapunov functions for time-varying systems satisfying generalized conditions of Matrosov theorem," *Math. Control Signals Syst.*, vol. 19, no. 2, pp. 151–182, 2007.print.

[16] H.L. Smith and P. Waltman, *The Theory of the Chemostat*. Cambridge: Cambridge University Press, 1995.print.

[17] E.D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Trans. Autom. Control*, vol. 34, no. 4, pp. 435–443, April 1989.print.

[18] E.D. Sontag, "Input-to-State Stability: Basic concepts and results," in *Nonlinear and Optimal Control Theory*, P. Nistri and G. Stefani, Eds. Berlin, Germany: Springer, 2008, pp. 163–220.print.

[19] J-P. Steyer, J-C. Bouvier, T. Conte, P. Gras, and P. Sousbie, "Evaluation of a four year experience with a fully instrumented anaerobic digestion process," *Water Science and Technology*, vol. 45, no. 4-5, pp. 495–502, 2002.print.