GENERALIZED DIFFERENTIAL TRANSFORM METHOD AND ITS APPLICATION TO SOLVE NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH SPACE AND TIME FRACTIONAL DERIVATIVES

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Abstract

In this paper a generalized FRDTM implemented to solve two dimensional time fractional heat and three dimensional fractional wave equations. Numerical efficiency of the proposed method have been depicted through graphical method including phase plot of II approximate solution and error diagram.

Keywords: *Time fractional differential equation; Fractional reduced differential transform method; Analytical approximate solution.*

2010 mathematics subject classification: 34A08, 35A20, 35A22.

1. INTRODUCTION AND PRELIMINARIES OF FRDTM

In the field of applied science and engineering like diffusion, electric circuits, fluid mechanics, relaxation processes etc., for the interpretation and mathematical modeling, wide use of Fractional order partial differential equations is accepted

Fractional modeling has significant contribution of several authors like S. L. Kalla 1969, Hilfer 2000, Klimek 2005, Kilbas 2006, Yang2016, Mianardi 2010 etc. [1-6]

Importance of fractional calculus is increased as it act as a powerful tool to elaborate physical complex problems for understanding the nature of matter and controlling design with no loss of hereditary behavior.

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Fractional Reduced Differential Transform method (FRDTM): Description of the method

Fractional reduced differential transform method (FRTDM) which is an analytical approximate method, is designed by Keskin and Oturanc [7] and it is used to solve large and difficult computation in many analytical methods.

If $u(x,t)$ be continuously differentiable function with respect to space variable x and time t such that

$$u(x,t) = h(x)g(t) \tag{1.1}$$

Then

$$u(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} U(i,j)x^i t^j \tag{1.2}$$

where $U(i,j) = h(i)g(i)$ is called the spectrum of $u(x,t)$.

Let $u(x,t)$ be analytic function then fractional reduced differential transform of u is given by (Srivastava et. al. 2006)[4]

$$U_k(x) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}u(x,t)}{\partial t^{k\alpha}}\right]_{t=t_0} \tag{1.3}$$

where α is the order of derivative which is taken as Caputo sense.

Further the inverse transform of $U_k(x)$ is given as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)(t-t_0)^{k\alpha} \tag{1.4}$$

Fractional Operations of FRDTM (Srivastava et. al., 2013)[1]

Let $u(x,t), v(x,t)$ and $w(x,t)$ are the analytic functions such as $u(x,t) = R_D^{-1}[U_k(x)]$, $v(x,t) = R_D^{-1}[V_k(x)]$ and $w(x,t) = R_D^{-1}[W_k(x)]$ then following properties holds

i. If $u(x,t) = v(x,t) \pm w(x,t)$, then $U_k(x) = V_k(x) \pm W_k(x)$

ii. If $u(x,t) = av(x,t)$, then $U_k(x) = aV_k(x)$ where a is any constant.

iii. If $u(x,t) = x^m t^n v(x,t)$, then $U_k(x) = V_{k-n}(x)$

iv. If $u(x,t) = v(x,t).w(x,t)$, then $U_k(x) = \sum_{r=0}^{k} V_r(x)W_{k-r}(x)$

v. If $u(x,t) = \frac{\partial^r}{\partial x^r}v(x,t)$, then $U_k(x) = \frac{\partial^r}{\partial x^r}V_k(x)$

vi. If $u(x,t) = \frac{\partial^{\gamma\alpha}}{\partial t^{\gamma\alpha}} v(x,t)$, then $U_k(x) = \frac{\Gamma(\alpha k+\alpha\gamma+1)}{\Gamma(\alpha k+1)} V_{k+\gamma}(x)$

vii. If $u(x,t) = v_1(x,t)v_2(x,t)v_3(x,t)$, then $U_k(x) = \sum_{\gamma=0}^{k} \sum_{i=0}^{\gamma} U_i(x) U_{\gamma-i} U_{k-\gamma}(x)$

viii. If $u(x,t) = v_1(x,t)v_2(x,t)v_3(x,t)v_4(x,t)$, then

$U_k(x) = \sum_{\gamma=0}^{k} \sum_{i=0}^{\gamma} \sum_{j=0}^{i} U_j(x) U_{i-j}(x) U_{\gamma-i}(x) U_{k-\gamma}(x)$

ix. If $u(x,t) = x^m t^n$, then $U_k(x) = x^m \delta(k-n)$, $\delta(k) = \begin{cases} 1; & k=0 \\ 0; & k \neq 0 \end{cases}$

2.TWO DIMENSIONAL HEAT FLOW

Consider the following two dimensional initial boundary value problem as heat like equation

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x,y,t) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad 0 < x, \ y < \pi, \quad t > 0, \quad 0 < \alpha \leq 1 \tag{2.1}$$

with boundary condition as

$$\left.\begin{aligned} U(0,y,t) &= U(\pi,y,t) = 0 \\ U(x,0,t) &= U(x,\pi,t) = 0 \end{aligned}\right\} \tag{2.2}$$

and the initial condition as

$$U(x,y,0) = (\sin x)(\sin y) \tag{2.3}$$

By using the FRDTM of equation (2.1), we have recurrence relation as

$$U_{k+1}(x,y) = \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+\alpha+1)} \left[\frac{\partial^2}{\partial x^2} U_k(x,y) + \frac{\partial^2}{\partial y^2} U_k(x,y) \right] \tag{2.4}$$

Put k=0, 1, 2… to get following values

$$U_0(x,y,t) = (\sin x)(\sin y)$$

$$U_1(x,y,t) = \frac{1}{\Gamma(\alpha+1)} (-\sin x \sin y - \sin x \sin y)$$

$$= \frac{-2}{\Gamma(\alpha+1)} \sin x \sin y$$

$$U_2(x,y,t) = \frac{(2)^2}{\Gamma(2\alpha+1)} \sin x \sin y$$

Similarly

$$U_3(x,y,t) = \frac{-(2)^3}{\Gamma(3\alpha+1)} \sin x \sin y$$

⋮

Thus, the approximation solution can be written by

$$U_n(x,y,t) = \frac{(-1)^n (2)^n}{\Gamma(n\alpha+1)} \sin x \sin y$$

Thus, the approximation solution of (2.1) can be written as

$$U(x,y,t) = \sin x \sin y - \frac{2}{\Gamma(\alpha+1)} \sin x \sin y \, t^{\alpha} + \frac{(2)^2}{\Gamma(2\alpha+1)} \sin x \sin y \, t^{2\alpha} - \frac{(2)^3}{\Gamma(3\alpha+1)} \sin x \sin y \, t^{3\alpha} + \quad \dots$$
(2.5)

Setting $\alpha = 1$, equation (2.5) can be written as

$$U(x,y,t) = e^{-2t} \sin x \sin y \qquad (2.6)$$

Figures

Figure

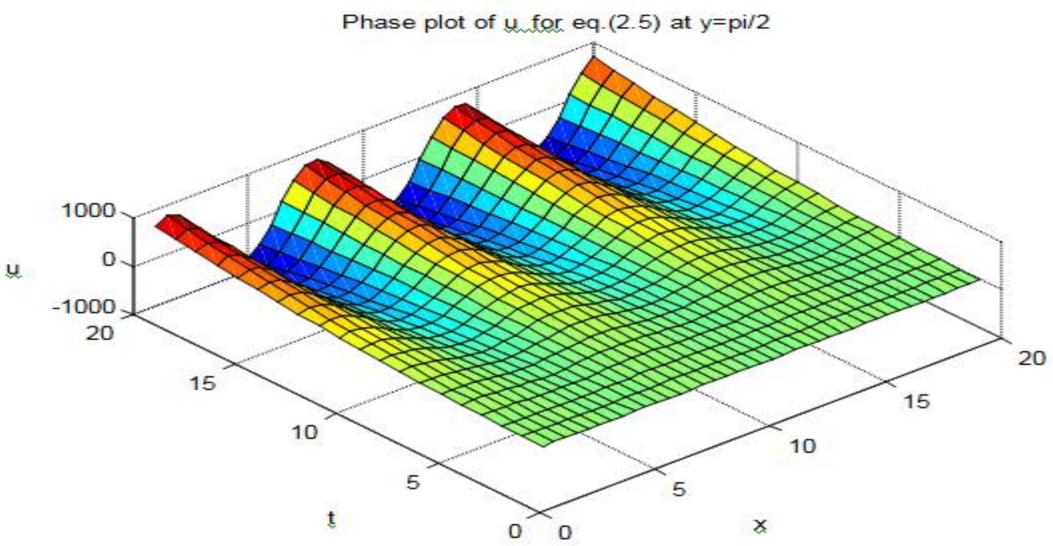

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Figure2

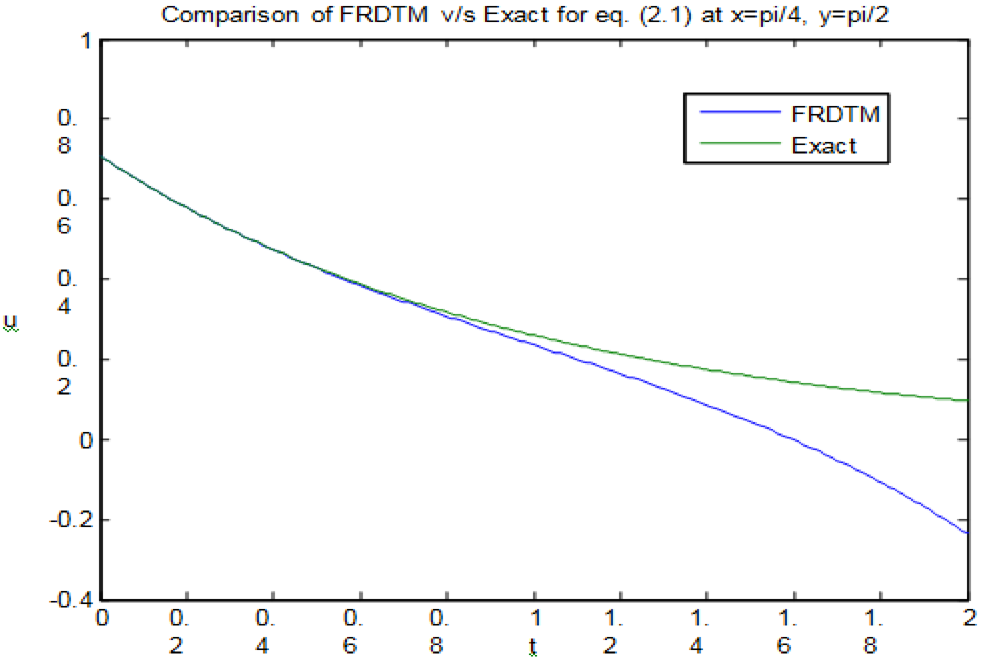


Figure3

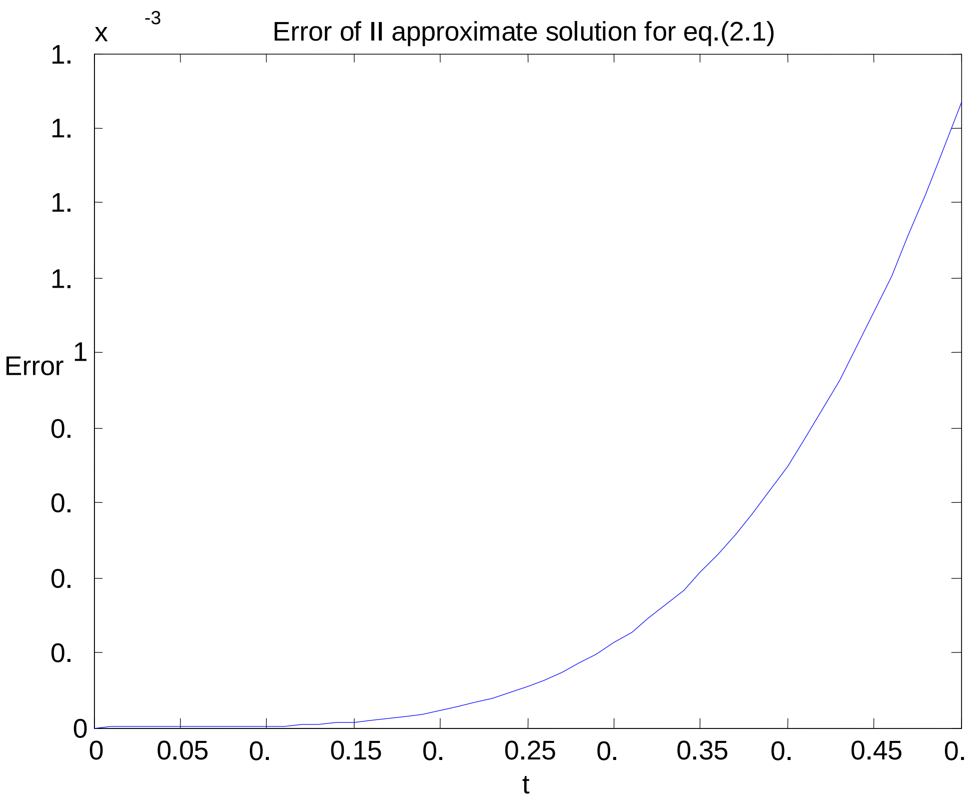
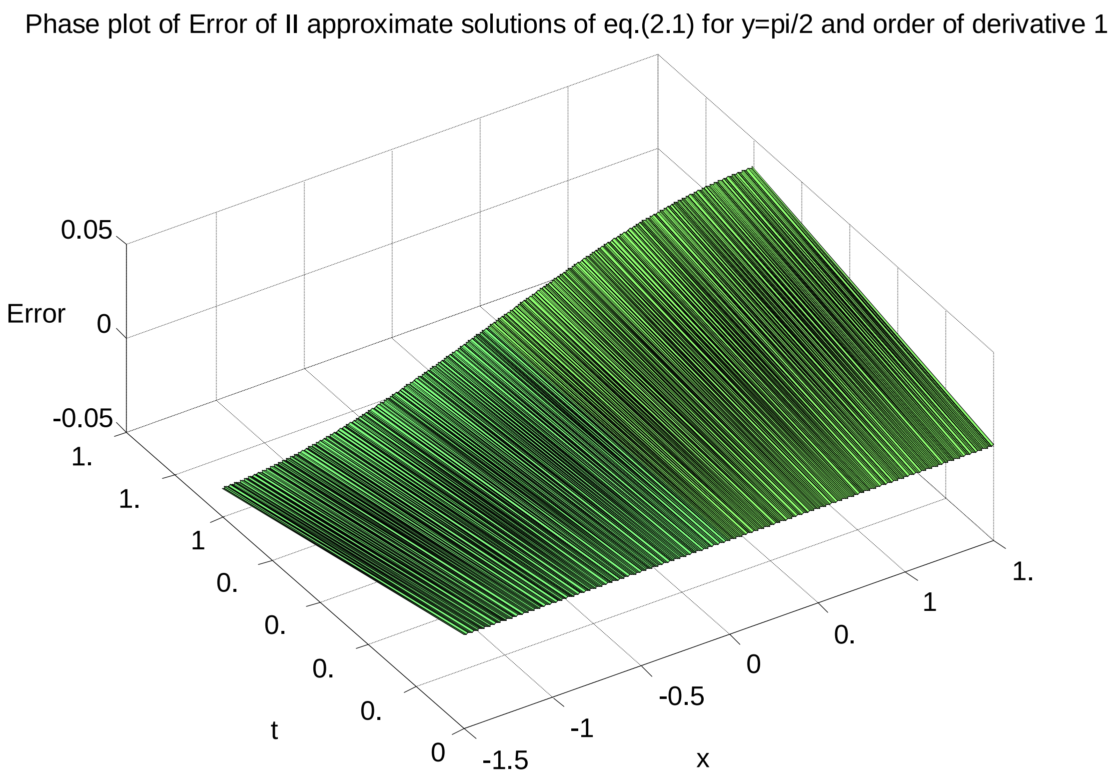


Figure4



3.Three dimensional fractional wave equation

The propagation of waves in a three dimensional volume of length a, width b, and height d is governed by the following initial boundary value problem

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} = 3\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \qquad t > 0, \quad 1 \le \alpha \le 2 \tag{3.1}$$

with the following boundary condition as

$$\left.\begin{array}{c} u(0,y,z,t) = u(a,y,z,t) = 0 \\ u(x,0,z,t) = u(x,b,z,t) = 0 \\ u(x,y,0,t) = u(x,y,d,t) = 0 \end{array}\right) \tag{3.2}$$

and the initial condition as

$$u(x,y,z,0) = 3\sin x \sin y \sin z, \qquad u_t(x,y,z,0) = 0 \tag{3.3}$$

By using the FRTDM of equation (3.1), we have recurrence relation as

$$U_{k+2}(x) = \frac{3\Gamma(\alpha k+1)}{\Gamma(\alpha k+2\alpha+1)}\left[\frac{\partial^2 U_k}{\partial x^2} + \frac{\partial^2 U_k}{\partial y^2} + \frac{\partial^2 U_k}{\partial z^2}\right] \tag{3.4}$$

With FRTDM of initial condition

$U_0(x) = 3\sin x \sin y \sin z$

$U_1(x) = 0$

By straight forward iteration yields

$$U_2(x) = \frac{3}{\Gamma(2\alpha+1)}[-9\sin x \sin y \sin z] = \frac{-27\sin x \sin y \sin z}{\Gamma(2\alpha+1)}$$

$$U_4(x) = \frac{3\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)}\left[\frac{-27(-3\sin x \sin y \sin z)}{\Gamma(2\alpha+1)}\right]$$

$$= \frac{3.3^4}{\Gamma(4\alpha+1)}\sin x \sin y \sin z$$

$\vdots$

Thus, the approximation series solution

$$U(x,y,z,t) = 3\sin x \sin y \sin z - \frac{3.3^2 t^{2\alpha}\sin x \sin y \sin z}{\Gamma(2\alpha+1)} + \frac{3.3^4\, t^{4\alpha}\sin x \sin y \sin z}{\Gamma(4\alpha+1)} - \cdots$$

$$U(x,y,z,t) = 3\sin x \sin y \sin z\left[1 - \frac{(3t^{\alpha})^2}{\Gamma(2\alpha+1)} + \frac{(3t^{\alpha})^4}{\Gamma(4\alpha+1)} - \right] \quad \ldots (3.5)$$

Setting$\alpha = 1$, equation (3.5) yields the exact solution

$U(x,y,z,t) = 3\sin x \sin y \sin z \cos 3t$

4.Conclusion

Efficient numerical methods for nonlinear fractional ordinary differential equations and their applications to solve mathematical models are currently under development will serve to demonstrate, evaluate and refine the research aims described above.

In the present paper the generalized FRDTM implemented to solve linear and non- linear two and three dimensional fractional differential equations. The generalized method gives better realistic series solutions which converge rapidly and results obtained just at second iteration are in excellent agreement with the exact results. The results reveal that solution continuously depends on time fractional derivatives and valid for long time in integer case. This generalized method was used directly without employing linearization and perturbation. The efficiency and capability of the present FRDTM have been checked via several illustrated examples. The results reveal the complete reliability of this method with a great potential in scientific applications. Finally, we may conclude that the FRDTM is very powerful, straightforward and effective to obtain analytical numerical solutions of a wide variety problems related to fractional PDEs applied in mathematics, physics and engineering.

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